International J. of Math. Sci. & Engg. Appls. (IJMSEA)
ISSN 0973-9424, Vol. 11 No. III(December, 2017), pp.1-16

DYNAMICS OF STABILITY AND HOPF BIFURCATION ANALYSIS OF A FIVE SPECIES STAGE-STRUCTURED SYSTEM

VIJAY KUMAR[1,2], JOYDIP DHAR[3], HARBAX SINGH BHATTI[4] AND HARKARAN SINGH[5]

[1] Beant College of Engineering and Technology, Gurdaspur-143521, Punjab, India
[2] I.K.G. Punjab Technical University, Kapurthala-144601, Punjab, India
[3] ABV-Indian Institute of Information Technology and Management, Gwalior-474015, M. P., India
[4] B.B.S.B. Engineering College, Fatehgarh Sahib, Punjab, India
[5] Khalsa College of Engineering Technology, Amritsar, Punjab, India

Abstract

In this study, dynamics of stability and Hopf bifurcation analysis of a five species stage-structured system is proposed. Boundedness and positivity of the system are studied. Stability analysis is carried out for all possible equilibrium states. Hopf bifurcation are studied and analyzed at all possible equilibrium points for the proposed system. The sensitivity analysis of the variables at the interior equilibrium w.r.t. the system parameters is performed and identified the respective sensitive indices of the variables. Further numerical simulations have been carried out to support our analytic results.

Key Words : *Stage-structured system, Boundedness and positivity, Hopf bifurcation, Sensitivity analysis, Numerical simulation.*

1. Introduction

A pest is a harmful insect for the plant, and its control has become a challenge in agriculture. It's outbreak often causes serious ecological and economical problems [1,2]. The technique of controlling the pests species is important if pests are extinct or reduce at acceptably low levels. For the progress of biological control methods, the dynamics of pest and its natural enemy populations should have to be developed. Therefore, to discuss farming problems, mathematical models are useful, since these enable the authors to have a general idea of the system. Mathematical models developed by introducing delay factors and functional responses, are of great importance and helps to improve the knowledge of complex phenoms. This means that the simplest mathematical systems can't grab, these variations of dynamics of the models [11,12]. The biological control strategy is the method of releasing natural enemies or infective pests for pest control, which is an emphasis area in recent years because of less damage to the environment than chemical methods. Biological control has proven relatively successful and safe. In population dynamics, pest and natural enemy interactions treat as prey-predator interactions and time delays have come to play an important role in prey-predator systems. Mankind has mostly used these models as tools to solve such problems and to understand the behavior of population systems. The model will, of course, could not contain all the features of the real system, but it is important that the model contains the characteristic features which are essential in the context of the problem to be solved or described up to the desired level. The population changes may have important economic and sociologic consequences; for example, the farmer wants to know how large the pest population in his crop is most vulnerable and what effects pesticide spraying will have? Therefore, our main focus will be on controlling pest populations through a prey-predator dynamics. In this context, many researchers have studied different types of predator-prey systems with single delay and analyzed the stability and bifurcation (e.g. periodic behavior) of the systems [3-5, 7, 14, 19]. Also a little attention has paid towards the stability and bifurcation of the prey-predator model with two delays as discussed in [8-10, 13]. A wide range of pest control strategies is available to the farmers and most methods are based on chemical insecticides. These chemical pesticides have many disadvantages [6,15]. Again, biological control suppresses the pest populations with the help of other living organisms, often called natural enemies or beneficial species [16-18].

Keeping in mind the above literature, in the present study, we proposed and analyzed a stage structured plant-pest-natural enemy food chain model. This paper is organized as follows: in section 2, formulation of the model is presented. In section 3, positivity and boundedness of the system have discussed. In section 4, the system is analyzed for the asymptotic stability at all possible four equilibrium states and obtained the conditions for the existence of Hopf bifurcation at the natural enemy free equilibrium, E_2 and the coexisting equilibrium point, E^*. In section 5 and 6, Hopf bifurcation analysis at E_2 and E^* are presented and analyzed respectively. In section 7, the sensitivity analysis of the state variables at coexisting equilibrium state with respect to system parameters is performed. In section 8, numerical simulations are carried out to support our analytical findings. Finally, the results have been discussed in the conclusion section.

2. The Mathematical Model

The assumptions of the proposed system are:

(i) In a particular patch, there are five types of populations, namely, plant $(X(t))$, immature pest $(P_1(t))$, mature pest $(P_2(t))$, immature natural enemy $(N_1(t))$ and mature natural enemy $(N_2(t))$.

(ii) The plant population grows logistically with α as the intrinsic growth rate and k being the carrying capacity. Thus, the per capita growth rate of the plant is $\alpha X\left(1 - \frac{X}{k}\right)$ in the absence of pests.

(iii) Mature pests harvest plants, and harvesting functional response is Holling type-I.

(iv) Mature pests can hide them self from the mature natural enemies, hence the mature natural enemy harvesting pests with Holling type-II response function.

(v) Let β be the predation rate of the plants by mature pests; γ be the harvesting rate of mature pests by the mature natural enemy. Let a be the half saturation constant; β_1 and γ_1 be the conversion rates for mature pest and mature natural enemy respectively. Let ξ and δ be the maturity rates for immature pest, $P_1(t)$ and immature natural enemy respectively, $N_1(t)$. Suppose δ_1 is the migration rate of mature natural enemy, $N_2(t)$ and ξ_1 is the overcrowding rate of mature pest. Also, let μ_1, μ_2, μ_3 and μ_4 be the natural mortality rates for immature pest, mature pest, immature and mature natural enemies respectively.

Keeping in view of the assumptions and interactions, our proposed system is of the form:

$$\frac{dX}{dt} = \alpha X\left(1 - \frac{X}{k}\right) - \beta X P_2, \tag{1}$$

$$\frac{dP_1}{dt} = \xi P_2 - \mu_1 P_1, \tag{2}$$

$$\frac{dP_2}{dt} = \beta_1 X P_2 - \frac{\gamma P_2 N_2}{a + P_2} - \mu_2 P_2 - \xi_1 P_2^2, \tag{3}$$

$$\frac{dN_1}{dt} = \delta N_2 - \mu_3 N_1, \tag{4}$$

$$\frac{dN_2}{dt} = \frac{\gamma_1 P_2 N_2}{a + P_2} - \delta_1 N_2 - \mu_4 N_2, \tag{5}$$

with initial conditions: $X(t) > 0$, $P_1(t) > 0$, $P_2(t) > 0$, $N_1(t) > 0$ and $N_2(t) > 0$ for all $t \geq 0$.

3. Positivity and Boundedness

In this section, we state and prove the following lemmas for the positivity and boundedness of the solution of the systems:

Lemma 3.1 : The solution of the system of equations (1)-(5) with positive initial populations, for all $t \geq 0$.

Proof : Let $((X(t), P_1(t), P_2(t), N_1(t), N_2(t))$ be a solution of the system (1) - (5) with positive initial populations. Let us consider $X(t)$ for $t \in [0, \tau]$. From the equation (1), we obtain

$$\frac{dX}{dt} \geq -\frac{\alpha X^2}{k} - \beta X P_2,$$

it follows that

$$X(t) \geq \frac{\exp\left\{-\int_0^t (\beta P_2)\, du\right\}}{X(0) + \int_0^t \frac{\alpha}{k} \exp\left\{-\int_0^t (\beta P_2)\, du\right\} dv} > 0.$$

The equation (2) can be written as,

$$\frac{dP_1}{dt} \geq -\mu_1 P_1,$$

which evidences that

$$P_1(t) \geq P_1(0)\exp\left\{-\int_0^t (\mu_1)du\right\} > 0.$$

Also for $t \in [0, \tau]$, the equation (3) can be represented as

$$\frac{dP_2}{dt} \geq -\frac{\gamma P_2 N_2}{a + P_2} - \mu_2 P_2 - \xi_1 P_2^2,$$

which evidences that

$$P_2(t) \geq \frac{\exp\left\{-\int_0^t (\mu_2)\, du\right\}}{P_2(0) + \int_0^t (\xi_1 + \frac{\gamma N_2}{P_2(a+P_2)})\exp\left\{-\int_0^t (\mu_2)\, du\right\} dv} > 0.$$

The equation (4) for $t \in [0, \tau]$ can be rewritten as

$$\frac{dN_1}{dt} \geq -\mu_3 N_1,$$

which evidences that

$$N_1(t) \geq N_1(0)\exp\left\{-\int_0^t (\mu_3)du\right\} > 0.$$

Also, the equation (5) for $t \in [0, \tau]$ can be written as

$$\frac{dN_2}{dt} \geq -(\delta_1 + \mu_4)N_2,$$

which evidences that

$$N_2(t) \geq N_2(0)\exp\left\{-\int_0^t (\delta_1 + \mu_4)du\right\} > 0.$$

In the similar way, we can treat the intervals $[\tau, 2\tau], ..., [n\tau, (n+1)\tau], n \in N$. Hence the proposed system is positive, i.e., $X(t) > 0$, $P_1(t) > 0$, $P_2(t) > 0$, $N_1(t) > 0$ and $N_2(t) > 0$ for all $t \geq 0$ by induction. □

Lemma 3.2 : The solution of the system of equations (1) - (5) with initial conditions is uniformly bounded in Ω, where

$$\Omega = \left\{(X, P_1, P_2, N_1, N_2) : 0 \leq X(t) + P_1(t) + P_2(t) + N_1(t) + N_2(t) \leq \frac{(\alpha + \mu')^2 k}{4\alpha\mu'}\right\},$$

$\mu' = min\{\mu_1, \mu_2, \mu_3, \mu_4\}$.

Proof : Let $W(t) = X(t) + P_1(t) + P_2(t) + N_1(t) + N_2(t)$. Taking the derivative of $W(t)$ with respect to t, we have

$$\frac{dW(t)}{dt} = \alpha X\left(1 - \frac{X}{k}\right) - \mu'(W - X) + \xi P_2 - \xi_1 P_2^2.$$

Since $\beta_1 << \beta$, $\gamma_1 << \gamma$, $\delta_1 << \delta$ and ξ is to be small, therefore we have

$$\frac{dW(t)}{dt} + \mu' W \quad \leq \quad \alpha X\left(1 - \frac{X}{k}\right) + \mu' X.$$

Taking $\mu' = min\{\mu_1, \mu_2, \mu_3, \mu_4\}$, we get

$$\frac{dW(t)}{dt} + \mu' W \leq \frac{(\alpha + \mu')^2)k}{4\alpha}.$$

Therefore,

$$0 \leq W(t) \leq W(0)e^{-u't} + \frac{(\alpha + \mu')^2 k}{4\alpha\mu'}.$$

As $t \to \infty$, we have

$$0 \leq W(t) \leq \frac{(\alpha + \mu')^2 k}{4\alpha\mu'}.$$

Hence, $W(t)$ is bounded, i.e., the proposed system is bounded. □

4. Equilibria and Stability Analysis of the System

The system of equations (1) - (5) have four feasible equilibrium points:

(i) The equilibrium point $E_0(0, 0, 0, 0, 0)$ always exists.

(ii) The equilibrium point $E_1(k, 0, 0, 0, 0)$ exists.

(iii) Natural enemy-free equilibrium $E_2\left(\frac{k(\beta\mu_2+\alpha\xi_1)}{k\beta\beta_1+\alpha\xi_1}, \frac{\alpha\xi(k\beta_1-\mu_2)}{\mu_1(k\beta\beta_1+\alpha\xi_1)}, \frac{\alpha(k\beta_1-\mu_2)}{k\beta\beta_1+\alpha\xi_1}, 0, 0\right)$ exists, if H_1 holds, where $H_1 : k\beta_1 > \mu_2$.

(iv) Interior equilibrium $E^*(X^*, P_1^*, P_2^*, N_1^*, N_2^*)$ exists only when $\gamma_1 > max$ $\left\{(\delta_1 + \mu_4)\left[\frac{a\beta+\alpha}{\alpha}, 1, \left(1 + \frac{a(k\beta\beta_1+\alpha\xi_1)}{\alpha(k\beta_1-\mu_2)}\right)\right]\right\}$, where X^*, P_1^*, P_2^*, N_1^*, N_2^* is given by

$$\begin{cases} X^* = k + \frac{ak\beta}{\alpha} + \frac{ak\beta\gamma_1}{\alpha(-\gamma_1+\delta_1+\mu_4)}, \\ P_1^* = -\frac{a\xi(\delta_1+\mu_4)}{\mu_1(-\gamma_1+\delta_1+\mu_4)}, \\ P_2^* = -\frac{a(\delta_1+\mu_4)}{-\gamma_1+\delta_1+\mu_4}, \\ N_1^* = -\frac{a\delta\gamma_1(\alpha\gamma_1\mu_2+k\beta_1(-\alpha\gamma_1+(\alpha+a\beta)(\delta_1+\mu_4))+\alpha(\delta_1+\mu_4)(-\mu_2+a\xi_1))}{\alpha\gamma\mu_3(-\gamma_1+\delta_1+\mu_4)^2}, \\ N_2^* = -\frac{a\gamma_1(\alpha\gamma_1\mu_2+k\beta_1(-\alpha\gamma_1+(\alpha+a\beta)(\delta_1+\mu_4))+\alpha(\delta_1+\mu_4)(-\mu_2+a\xi_1))}{\alpha\gamma(-\gamma_1+\delta_1+\mu_4)^2}. \end{cases} \tag{6}$$

Now, we are going to discuss the local behavior of positive equilibria of the system (1) - (5),

Theorem 4.1 : The local behavior of different equilibria of the system (1) - (5) is as follows;

(i) $E_0(0,0,0,0,0)$ is always unstable.

(ii) $E_1(k,0,0,0,0)$ is stable only when $\mu_2 > k\beta_1$.

Proof :

(i) The characteristic equation for $E_0(0,0,0,0,0)$ is

$$(-\lambda+\alpha)(-\lambda-\mu_1)(-\lambda-\mu_2)(-\lambda-\mu_3)(-\lambda-(\delta_1+\mu_4))=0. \tag{7}$$

The eigenvalues are $\lambda=\alpha$, $\lambda=-\mu_1$, $\lambda=-\mu_2$, $\lambda=-\mu_3$ and $\lambda=-(\delta_1+\mu_4)$. The equilibrium $E_0(0,0,0,0,0)$ is always unstable, because one of the eigenvalues of (7) is positive.

(ii) The characteristic equation for $E_1(k,0,0,0,0)$ is

$$(-\lambda-\alpha)(-\lambda-\mu_1)(-\lambda-\mu_3)(-\lambda-(\delta_1+\mu_4))(-\lambda+k\beta_1-\mu_2)=0. \tag{8}$$

The eigenvalues are $\lambda=-\alpha$, $\lambda=-\mu_1$, $\lambda=-\mu_3$, $\lambda=-(\delta_1+\mu_4)$ and $\lambda=k\beta_1-\mu_2$. It is observed that the equilibrium point $E_1(k,0,0,0,0)$ is stable only when $\mu_2 > k\beta_1$. □

Theorem 4.2 : For the system (1) - (5),

(i) The natural enemy free equilibrium point E_2 is locally asymptotically stable, because two eigen values of the characteristic equation,i.e.,

$$\lambda^3+A_{111}\lambda^2+A_{222}\lambda+A_{333}=0, \tag{9}$$

at E_2 are directly negative, i.e., $\lambda=-\mu_1$ and $\lambda=-\mu_3$. For remaining three eigen values, the Routh-Hurwitz criterion is satisfied, i.e., $A_{iii}>0$, $i=1,2,3$ and $A_{111}A_{222}-A_{333}>0$ holds, where $A_{111}=-(a_1+a_4+a_6)$, A_{222}=$a_1a_4-a_2a_3+a_6(a_1+a_4)$, A_{333}=$a_6(a_2a_3-a_1a_4)$; $a_1=-\frac{2X\alpha}{k}+\alpha-P_2\beta$, $a_2=-X\beta$, $a_3=P_2\beta_1$, $a_4=X\beta_1-\mu_2-2\xi_1P_2$, $a_5=-\frac{P_2\gamma}{a+P_2}$, $a_6=\frac{P_2\gamma_1}{a+P_2}-\delta_1-\mu_4$; $X=\frac{k(\beta\mu_2+\alpha\xi_1)}{k\beta\beta_1+\alpha\xi_1}$, $P_2=\frac{\alpha(k\beta_1-\mu_2)}{k\beta\beta_1+\alpha\xi_1}$. Also it is demonstrated graphically see Figure 2.

(ii) The natural enemy free equilibrium point E_2 is unstable otherwise and the system undergoes Hopf bifurcation around E_2, for graphical representation see Figure 3.

5. Hopf-bifurcation Analysis at E_2

Further, we will study the Hopf-bifurcation of the system (1) - (5), taking "β" (i.e., predation rate of plants by mature pests as a bifurcation parameter). Now, the necessary and sufficient condition for the existence of the Hopf-bifurcation, if there exists $\beta = \beta_0$ such that (i) $= A_{iii}(\beta_0) > 0$, $i = 1, 2, 3$, (ii) $= A_{111}(\beta_0)A_{222}(\beta_0) - A_{333}(\beta_0) = 0$ and (iii) if we consider the eigen values of the characteristic equation at E_2 of the system of the form $\lambda_i = u_i + iv_i$, then $Re\frac{d}{d\beta}(u_i) \neq 0$, $i = 1, 2, 3$. After putting the values, the condition $A_{111}A_{222} - A_{333} = 0$ becomes

$$v_{11}\beta^5 + v_{22}\beta^4 + v_{33}\beta^3 + v_{44}\beta^2 + v_{55}\beta + v_{66} = 0, \tag{10}$$

where $v_{11} = (e_4 + e_3 + f_3h_3)$, $v_{22} = (e_5 + e_2 + f_3h_4 + f_4h_3)$, $v_{33} = (e_6 + e_1 + f_3h_5 + f_4h_4 + h_3f_5)$, $v_{44} = (e_7 + f_9 + f_4h_5 + h_4f_5 + h_3f_6)$, $v_{55} = (e_8 + f_8 + f_5h_5 + h_4f_6)$, $v_{66} = (e_9 + f_7 + f_6h_5)$, $e_1 = g_7(h_1h_7 + h_2h_8 + h_6g_9) + g_8(h_2h_7 + h_1h_6)$, $e_2 = g_7(h_2h_7 + h_1h_6) + g_8h_2h_6$, $e_3 = h_2h_6g_7$, $e_4 = -g_1g_2g_6h_3$, $e_5 = -g_1g_2g_6h_4 - h_3(g_1g_3g_6 + g_1g_2g_5)$, $e_6 = -(g_1g_2g_6h_5 - h_4(g_1g_3g_6 + g_1g_2g_5) + h_3(g_1g_3g_5 + g_1g_4g_6))$, $e_7 = -(h_5(g_1g_3g_6 + g_1g_2g_5) + h_4(g_1g_3g_5 + g_1g_4g_6) + h_3g_1g_4g_5)$, $e_8 = -(h_5(g_1g_3g_5 + g_1g_4g_6) + h_4g_1g_4g_5)$, $e_9 = -h_5g_1g_4g_5$, $g_1 = k\beta_1 - \mu_2$, $g_2 = k\beta_1\mu_2$, $g_3 = \alpha\xi_1(k\beta_1 + \mu_2)$, $g_4 = \alpha\xi_1$, $g_5 = \alpha\gamma_1\mu_2 - k\beta_1(\alpha\gamma_1 - \alpha(\delta_1 + \mu_4)) + \alpha(\delta_1 + \mu_4)(-\mu_2 + a\xi_1)$, $g_6 = ka\beta_1(\delta_1 + \mu_4)$, $g_7 = ka\beta_1$, $g_8 = \alpha(g_1 + a\xi_1)$, $g_9 = (\alpha + \delta_1 + \mu_4 + g_1)g_4ag_8 - \alpha\gamma_1ag_1g_4$, $h_1 = (\alpha + \delta_1 + \mu_4 + g_1)g_4ag_7 + \alpha + \delta_1 + \mu_4 + g_1kg_7g_8 - a(\alpha + k\beta_1)g_1g_8 - \alpha\gamma_1g_1kg_7$, $h_2 = (\alpha + \delta_1 + \mu_4 + g_1)kg_7^2 - a(\alpha + k\beta_1)g_1g_7$, $h_3 = kg_7^2$, $h_4 = ag_4g_7 + kg_7g_8$, $h_5 = ag_4g_8$, $h_6 = k\beta_1g_1\mu_2$, $h_7 = g_4k\beta_1g_1 + g_1g_4\mu_2$, $h_8 = g_1g_4^2$, $h_9 = k\beta_1\xi_1(\alpha + g_1) + g_4\mu_2$, $f_1 = g_4(\alpha + g_1)\xi_1$, $f_2 = (g_1 + a\xi_1)\alpha(\delta_1 + \mu_4) - g_1\alpha\gamma_1)$, $f_3 = g_2g_6$, $f_4 = g_2f_2 + g_6h_9$, $f_5 = f_2h_2 + g_6f_1$, $f_6 = f_1f_2$, $f_7 = g_9g_8h_8$, $f_8 = g_9h_8g_7 + g_8(h_1h_8 + h_7g_9)$, $f_9 = g_7(h_1h_8 + h_7g_9) + g_8(h_1h_7 + h_2h_8 + h_6g_9)$. For example, taking parametric values: $\alpha = 0.05$; $k = 10$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.05$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$. we get a positive root $\beta = 0.21$ of equation (10). Therefore, the eigen values of the characteristic equation at E_2 of the system are of the form $\lambda_{1,2} = \pm iv$ and $\lambda_3 = -w$, where v and w are positive real numbers. Now we will verify the condition (iii) of Hopf-bifurcation. Put $\lambda = u + iv$ in equation (9), we have

$$(u+iv)^3 + A_{111}(u+iv)^2 + A_{222}(u+iv) + A_{333} = 0. \tag{11}$$

On separating the real and imaginary part and eliminating v between real and imaginary part, we get

$$8u^3 + 8A_{111}u^2 + 2(A_{111}^2 + A_{222})u + A_{111}A_{222} - A_{333} = 0. \tag{12}$$

It is clear from the above equation (12) that $u(\beta_0) = 0$ iff $A_{111}(\beta_0)A_{222}(\beta_0) - A_{333}(\beta_0) = 0$. Further, at $\beta = \beta_0$, $u(\beta_0)$ is the only root, since the discriminant of $8u^2 + 8A_{111}u + 2(A_{111}^2 + A_{222}) = 0$ is $64A_{111}^2 - 64(A_{111}^2 + A_{222}) < 0$. Again, differentiating (12) w.r.t. β, we have, $(24u^2 + 16A_{111}u + 2(A_{111}^2 + A_{222}))\frac{du}{d\beta} + (8u^2 + 4A_{111}u)\frac{dA_{111}}{d\beta} + 2u\frac{dA_{222}}{d\beta} + \frac{d}{d\beta}(A_{111}A_{222} - A_{333}) = 0$. Now, since at $\beta = \beta_0$, $u(\beta_0) = 0$, we get, $[\frac{du}{d\beta}]_{\beta=\beta_0} = \frac{-\frac{d}{d\beta}(A_{111}A_{222} - A_{333})}{2(A_{111}^2 + A_{222})} \neq 0$, which will ensure that the above system has a Hopf-bifurcation at E_2. And it is shown graphically in the Figure 3.

Theorem 5.1 : For the system (1) - (5),

(i) The interior equilibrium point E^* is locally asymptotically stable, because two eigen values of the characteristic equation,i.e.,

$$\lambda^3 + A_{1111}\lambda^2 + A_{2222}\lambda + A_{3333} = 0, \tag{13}$$

at E^* are $\lambda = -\mu_1$ and $\lambda = -\mu_3$. For remaining three eigen values, the Routh-Hurwitz criterion is satisfied, i.e., $A_{iiii} > 0$, $i = 1,2,3$ and $A_{1111}A_{2222} - A_{3333} > 0$ holds, where $A_{1111} = a_2b_1 - a_1 - b_2 - d_1$, $A_{2222}{=}a_1b_2 + a_1d_1 + d_1b_2 - a_1a_2b_1 - c_1c_2 - a_2b_1d_1$, $A_{3333}{=}a_1(c_1c_2 + a_2b_1d_1 - d_1b_2)$; $a_1 = -\frac{2X^*\alpha}{k} + \alpha - P_2^*\beta$, $a_2 = -X^*\beta$, $b_1 = P_2^*\beta_1$, $b_2 = -\frac{aN_2^*\gamma}{(a+P_2^*)^2} + X^*\beta_1 - \mu_2 - 2\xi_1P_2^*$, $c_1 = -\frac{P_2^*\gamma}{a+P_2^*}$, $c_2 = \frac{aN_2^*\gamma_1}{(a+P_2^*)^2}$, $d_1 = \frac{P_2^*\gamma_1}{a+P_2^*} - \delta_1 - \mu_4$; $X^* = k + \frac{ak\beta}{\alpha} + \frac{ak\beta\gamma_1}{\alpha(-\gamma_1+\delta_1+\mu_4)}$, $P_2^* = -\frac{a(\delta_1+\mu_4)}{-\gamma_1+\delta_1+\mu_4}$, $N_2^* = -\frac{a\gamma_1(\alpha\gamma_1\mu_2 + k\beta_1(-\alpha\gamma_1 + (\alpha+a\beta)(\delta_1+\mu_4)) + \alpha(\delta_1+\mu_4)(-\mu_2+a\xi_1))}{\alpha\gamma(-\gamma_1+\delta_1+\mu_4)^2}$. Also it is demonstrated graphically see Figure 4.

(ii) The interior equilibrium point E^* is unstable otherwise and the system undergoes Hopf bifurcation around E^*, for graphical representation see Figure 5.

6. Hopf-bifurcation Analysis at E^*

Further, we will study the Hopf-bifurcation of the system (1) - (5), taking "k" (i.e., carrying capacity as a bifurcation parameter). Now, the necessary and sufficient condition for the existence of the Hopf-bifurcation, if there exists $k = k_0$ such that (i) $= A_{iiii}(k_0) > 0$, $i = 1, 2, 3$, (ii) $= A_{1111}(k_0)A_{2222}(k_0) - A_{3333}(k_0) = 0$ and (iii) if we consider the eigen values of the characteristic equation at E^* of the system of the form $\lambda_i = u_i + iv_i$, then $Re\frac{d}{dk}(u_i) \neq 0$, $i = 1, 2, 3$. After putting the values, the condition $A_{1111}A_{2222} - A_{3333} = 0$ becomes

$$a_{11}k^2 + a_{22}k + a_{33} = 0, \tag{14}$$

where $a_{11} = m_1\ell_3$, $a_{22} = \ell_3(m_2 + m_3 + m_5)$, $a_{33} = p_2 + m_5(m_2 + m_3 + m_4)$, $m_1 = -\frac{(\ell_1\beta_1 p_2)(p_1^3 - \gamma_1 p_1^2 - a\beta\gamma_1\ell_1 - \ell_1 p_1)}{\gamma_1 p_1^3}$, $m_2 = \frac{p_2\alpha\ell_1(\alpha\gamma_1\mu_2 + \ell_2)}{p_1}$, $m_3 = -\frac{p_2(\alpha\gamma_1\mu_2 + \ell_2)}{\gamma_1}$, $m_4 = \frac{p_2(\ell_1\ell_2 + \alpha\ell_1\gamma_1(\mu_2 + a\xi_1))}{\gamma_1 p_1^2}$, $m_5 = \alpha^2\gamma_1^3 p_1^2 + \ell_4 - \ell_5 - \ell_6 - p_3 - p_4$, $\ell_1 = \alpha\gamma_1 - (\alpha + a\beta)(\delta_1 + \mu_4)$, $\ell_2 = \alpha(\delta_1 + \mu_4)(-\mu_2 + a\xi_1)$, $\ell_3 = (a\beta(1 + a\beta) + \alpha(2 + a\beta))\beta_1\gamma_1 p_1^2(\delta_1 + \mu_4)^2$, $\ell_4 = \alpha\gamma_1(\alpha + a\beta - 2\mu_2)p_1^2(\delta_1 + \mu_4)^2$, $\ell_5 = -\alpha\gamma_1^2 p_1^2(\delta_1 + \mu_4)(2\alpha + a\beta - \mu_2 - a\xi_1)$, $\ell_6 = -k\alpha\gamma_1^2 p_1^2(\delta_1 + \mu_4)(1 + a\beta)\beta_1$, $p_1 = \gamma_1 - \delta_1 - \mu_4$, $p_2 = \frac{\delta_1 + \mu_4}{\alpha^2\gamma_1}$, $p_3 = -p_1^2(\delta_1 + \mu_4)^2(\alpha + a\beta)\beta_1$, $p_4 = -p_1^2\alpha(\delta_1 + \mu_4)^2(-\mu_2 + a\xi_1)$. For example, taking parametric values: $\alpha = 0.6$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$; we get a positive root $k = 4.8$ of equation (14). Therefore, the eigen values of the characteristic equation at E^* of the system are of the form $\lambda_{1,2} = \pm iv$ and $\lambda_3 = -w$, where v and w are positive real numbers. Now we will verify the condition (iii) of Hopf-bifurcation. Put $\lambda = u + iv$ in equation (13), we have

$$(u + iv)^3 + A_{1111}(u + iv)^2 + A_{2222}(u + iv) + A_{3333} = 0. \tag{15}$$

On separating the real and imaginary part and eliminating v between real and imaginary part, we get

$$8u^3 + 8A_{1111}u^2 + 2(A_{1111}^2 + A_{2222})u + A_{1111}A_{2222} - A_{3333} = 0. \tag{16}$$

It is clear from the above equation (16) that $u(k_0) = 0$ iff $A_{1111}(k_0)A_{2222}(k_0) - A_{3333}(k_0) = 0$. Further, at $k = k_0$, $u(k_0)$ is the only root, since the discriminant of $8u^2 + 8A_{1111}u + 2(A_{1111}^2 + A_{2222}) = 0$ is $64A_{1111}^2 - 64(A_{1111}^2 + A_{2222}) < 0$. Again, differentiating (16) w.r.t. k, we have, $(24u^2 + 16A_{1111}u + 2(A_{1111}^2 + A_{2222}))\frac{du}{dk} + (8u^2 + 4A_{1111}u)\frac{dA_{1111}}{dk} +$

$2u\frac{dA_{2222}}{dk} + \frac{d}{dk}(A_{1111}A_{2222} - A_{3333}) = 0$. Now, since at $k = k_0$, $u(k_0) = 0$, we get, $[\frac{du}{dk}]_{k=k_0} = \frac{-\frac{d}{dk}(A_{1111}A_{2222}-A_{3333})}{2(A_{1111}^2+A_{2222})} \neq 0$, which will ensure that the above system has a Hopf-bifurcation at E^*. And it is shown graphically in the Figure 5.

7. Sensitivity Analysis

In this section, the sensitivity analysis of state variables of the system (1) - (5) w.r.t. system parameters at the interior equilibrium point is carried out. The respective sensitive parameters of the state variables at interior equilibrium are shown in the Table 1 using parameter values: $\alpha = 0.6$; $k = 1.8$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$. It is clear that α, k, γ_1 have a positive impact on X^*. Whereas β, a, δ_1, μ_4 have a negative impact on X^* and the rest of the parameters have a zero impact on the X^*. Moreover k is the most sensitive parameter to X^*. Also ξ, a, δ_1, μ_4 have a positive impact on P_1^*; γ_1 has a negative impact on the P_1^* as well as more sensitive to P_1^* and the rest of the parameters have no impact on P_1^*. Again we can see that a, δ_1, μ_4 have a positive impact on P_2^*; γ_1 has a negative impact on P_2^* as well as it is more sensitive parameter to P_2^* and the rest of parameters have zero impact on P_2^*. The parameters α, k, β_1, δ, γ_1 have positive impact on N_1^*; whereas the parameters β, γ, a, μ_2, ξ_1, μ_3, δ_1, μ_4 have a negative impact on N_1^* and the remaining parameters have zero impact on N_1^*. The parameters k and β_1 are more sensitive to N_1^*. Also, we see that the parameters α, k, γ_1, β_1 have positive impact on N_2^*; β, γ, a, μ_2, ξ_1, δ_1, μ_4 have a negative impact on N_2^* and the rest of parameters have zero impact on N_2^*. The parameters k and β_1 are more sensitive to N_2^*.

Table 1 : The sensitivity indices $\gamma_{y_j}^{x_i} = \frac{\partial x_i}{\partial y_j} \times \frac{y_j}{x_i}$ of the state variables of the system (1) - (5) to the parameters y_j for the parameter values: $\alpha = 0.6$; $k = 1.8$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$.

Parameter (y_j)	$\gamma_{y_j}^{X^*}$	$\gamma_{y_j}^{P_1^*}$	$\gamma_{y_j}^{P_2^*}$	$\gamma_{y_j}^{N_1^*}$	$\gamma_{y_j}^{N_2^*}$
α	0.03846	0	0	2.2499	2.2499
k	1	0	0	58.500	58.500
β	-0.03846	0	0	-2.2500	-2.2500
ξ	0	1	0	0	0

Parameter (y_j)	$\gamma_{y_j}^{X^*}$	$\gamma_{y_j}^{P_1^*}$	$\gamma_{y_j}^{P_2^*}$	$\gamma_{y_j}^{N_1^*}$	$\gamma_{y_j}^{N_2^*}$
μ_1	0	0	0	0	0
β_1	0	0	0	58.500	58.500
γ	0	0	0	-1	-1
a	-0.03846	1	1	-2.5000	-2.5000
μ_2	0	0	0	-56.250	-56.250
ξ_1	0	0	0	-1.2500	-1.2500
δ	0	0	0	1	0
μ_3	0	0	0	-1	0
γ_1	0.04274	-1.1111	-1.1111	3.7777	3.7777
δ_1	-0.02849	0.74074	0.74074	-2.5185	-2.5185
μ_4	-0.01425	0.37037	0.37037	-1.2592	-1.2592

8. Numerical Simulations

Numerical simulations of the system (1) - (5) are carried out to justify our analytic findings using MATLAB. The boundary equilibrium $E_1(5, 0, 0, 0, 0)$ is stable for parameter values: $\alpha = 0.8$; $k = 5$; $\beta = 0.03$; $\xi = 0.2$; $\mu_1 = 0.02$; $\beta_1 = 0.003$; $\gamma = 0.06$; $a = 1$; $\mu_2 = 0.2$; $\xi_1 = 0.05$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.03$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and result is shown in Figure 1. The equilibrium point $E_2(1.02, 0.00898, 0.0898, 0, 0)$ is stable for the parametric values: $\alpha = 0.05$; $k = 10$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.05$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $\beta = 0.5$, as shown in Figure 2 and the equilibrium is unstable and Hopf bifurcation appears for $\alpha = 0.05$; $k = 10$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.05$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $\beta = 0.21$, see the Figure 3. The interior equilibrium point $E^*(1.73, 0.011, 0.11, 0.0066, 0.0016)$ is stable for the parametric values: $\alpha = 0.6$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $k = 1.8$, see the result in Figure 4. The interior equilibrium point $E^*(1.73, 0.011, 0.11, 0.0066, 0.0016)$ is unstable and Hopf bifurcation appears for the parametric values: $\alpha = 0.6$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $k = 4.8$, see the result in Figure 5.

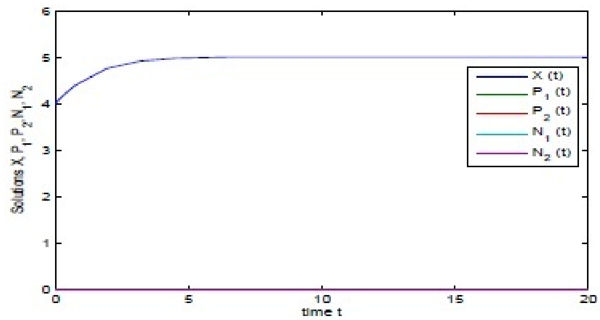

Figure 1: The boundary equilibrium $E_1(5, 0, 0, 0, 0)$ is stable for parameter values: $\alpha = 0.8$; $k = 5$; $\beta = 0.03$; $\xi = 0.2$; $\mu_1 = 0.02$; $\beta_1 = 0.003$; $\gamma = 0.06$; $a = 1$; $\mu_2 = 0.2$; $\xi_1 = 0.05$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.03$; $\delta_1 = 0.02$; $\mu_4 = 0.01$.

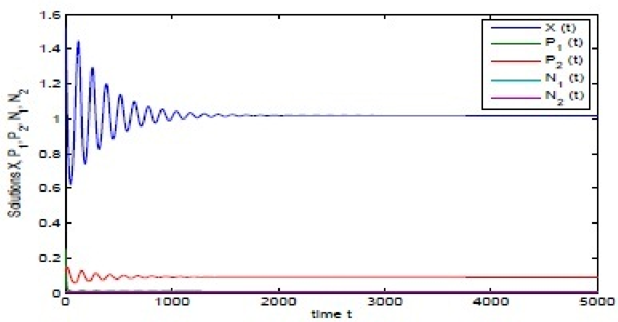

Figure 2: The equilibrium point $E_2(1.02, 0.00898, 0.0898, 0, 0)$ is stable for the parametric values: $\alpha = 0.05$; $k = 10$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.05$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $\beta = 0.5$ (bifurcation parameter).

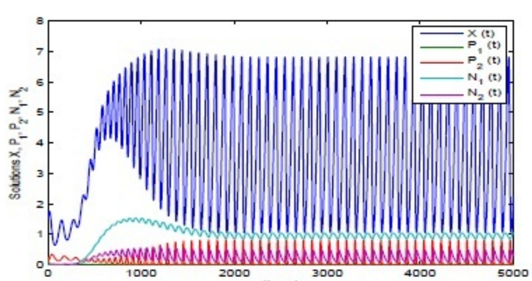

Figure 3: The equilibrium point $E_2(1.02, 0.00898, 0.0898, 0, 0)$ is unstable and Hopf bifurcation appears for parametric values:$\alpha = 0.05$; $k = 10$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.05$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $\beta = 0.21$ (bifurcation parameter).

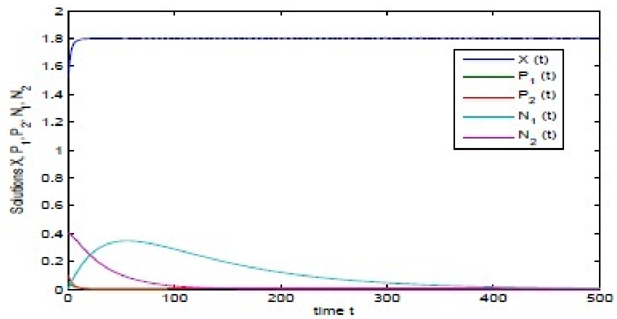

Figure 4: The interior equilibrium point $E^*(1.73, 0.011, 0.11, 0.0066, 0.0016)$ is stable for the parametric values: $\alpha = 0.6$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $k = 1.8$ (bifurcation parameter).

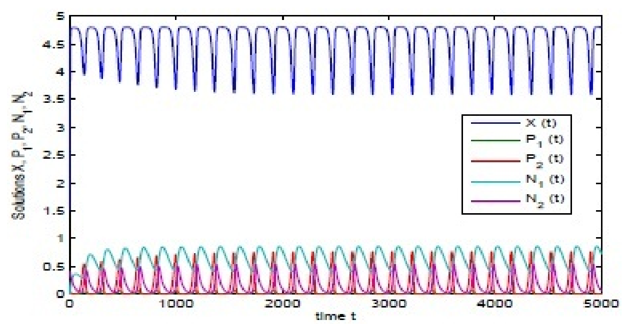

Figure 5: The interior equilibrium point $E^*(1.73, 0.011, 0.11, 0.0066, 0.0016)$ is unstable and Hopf bifurcation appears for the parametric values: $\alpha = 0.6$; $\beta = 0.2$; $\xi = 0.02$; $\mu_1 = 0.2$; $\beta_1 = 0.03$; $\gamma = 0.6$; $a = 1$; $\mu_2 = 0.05$; $\xi_1 = 0.01$; $\delta = 0.04$; $\mu_3 = 0.01$; $\gamma_1 = 0.3$; $\delta_1 = 0.02$; $\mu_4 = 0.01$ and $k = 4.8$ (bifurcation parameter).

9. Conclusion

In this paper, a plant-pest-natural enemy stage structured system is proposed. There are four possible steady states, and asymptotic stability of the system is studied for all equilibrium points. The existence of Hopf bifurcation at the natural enemy-free equilibrium, E_2 and interior equilibrium, E^* is explored. It is verified that the natural enemy free equilibrium E_2 is stable $\beta \geq 0.5$ and unstable when $\beta = 0.21$. Also, the interior equilibrium $E^*$is stable $k \leq 1.8$ and unstable when $k = 4.8$. Finally, the normalized forward sensitivity indices are determined for state variables at the interior equilibrium w.r.t. the system parameters. Simulations of the system are performed with a particular set of parameters to justify our analytic findings.

Acknowledgement

I would like to thanks, I. K. G. Punjab Technical University, Jalandhar-Kapurthala-144601, Punjab, India.

References

[1] Kishimba M. A., Henry L., Mwevura H., Mmochi A. J., Mihale M., Hellar H., The status of pesticide pollution in Tanzania, Talanta, 64(1) (2004), 48-53.

[2] Weaver R. D., Evans D. J., Luloff A. E., Pesticide use in tomato production: Consumer concerns and willingness-to-pay, Agribusiness, 8(2) (1992), 131-142.

[3] Lian F., Xu Y., Hopf bifurcation analysis of a predator-prey system with Holling type IV functional response and time delay, Applied Mathematics and Computation, 215(4) (2009), 1484-1495.

[4] Liu X., Han M., Chaos and Hopf Bifurcation analysis for a two species predator–prey system with prey refuge and diffusion, Nonlinear Analysis:Real World Applications, 12(2) (2011), 1047-1061.

[5] Song Y., Wei J., Local Hopf Bifurcation and global periodic soltions in a delayed predator prey system, Journal of Mathematical Analysis and Applications, 301(1) (2005), 1-21.

[6] Jatav K. S., Dhar J., Hybrid approach for pest control with impulsive releasing of natural enemies and chemical pesticides: A plant-Natural Enemy Model, Nonlinear Anal., Hybrid syst., 12 (2014), 79-92.

[7] Zhao H., Lin Y., Hopf bifurcation in a partial dependent predator-prey system with delay, Chaos, Solitons and Fractals, 42(2) (2009), 896-900.

[8] Faria T., Stability and bifurcation for a delayed predator-prey model and effect of diffusion, Journal of Mathematical Analysis and Applications, 254(2) (2001), 433-464.

[9] Li K., Wei J., Stability and Hopf bifurcation analysis of a prey-predator system with two delays, Chaos, Solitons and Fractals, 42(5) (2009), 2606-2613.

[10] Song Y., Peng Y., Wei J., Bifurcations for a predator-prey system with two delays, Journal of Mathematical Analysis and Applications, 337(1) (2008), 466-479.

[11] Holling C. S., Chattopadhyay J., The functional responses of predators to prey density and its role in mimicry and population dynamics, Mem. Entomol. Soc. Can., 97(45) (1965), 1-60.

[12] Singh H., Dhar J., Bhatti H. S., An epidemic model of childhood disease dynamics with maturation delay and latent period of infection, Modeling Earth Systems and Environment, 2(79) (2016).

[13] Xu C., Liao M., He X., Stability and Hopf bifurcation analysis for a Lotka-Volterra predator-prey model with two delays, International Journal of Applied Mathematics and Computer Science, 21(1) (2011), 97-107.

[14] Zhang L., Teng Z., Liu Z., Survival analysis for a periodic predator–prey model with prey impulsively unilateral diffusion in two patches, Applied Mathematical Modelling, 35 (2011), 4243-4256.

[15] Gazzoni D., Manejo de pragas da soja: uma abordagem hist Asrica, Londrina: EMBRAPACNPSO, (1994).

[16] Debach P., Rosen D., Biological Control by Natural Enemies, Cambridge University Press, (1991).

[17] Luff M., The potential of predators for pest control, Agriculture, ecosystems and environment, 10(2) (1983). 159-181.

[18] Rutledge C., Neil R.O'., Fox T., Landis D., Soyabean aphid predators and their use in integrated pest managemnet. Annals of the Entomological Society of America, 97(2) (2004), 240-248.

[19] Kumar V., Dhar J., Bhatti H.S., Singh H., Plant-pest-natural enemy dynamics with disease in pest and gestation delay for natural enemy, J. Math. Comput. Sci., 7(5) (2017), 948-965.