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STOCHASTIC BEHAVIOUR OF WATER PURIFICATION SYSTEM

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Abstract

In this paper, the author have considered an Water Purification system to analysis its stochastic behaviour. Supplementary variables have been used to convert the Non-Markovian process into Markovian. Laplace transform has been utilized to solve the mathematical model of considered system. Laplace transform of all transition state probabilities, steady-state behaviour of the system, availability and cost function of considered system have been obtained. A particular case has also been computed to enhance practical utility of the model. Graphical illustration followed by a numerical example has been appended in the end to highlight important results of the study.

1. Introduction

In the considered system, there are four main subsystems namely, Municipality Supply line, Tank, water purifier and Tap. The author has been taken one parallel redundant tap to enhance systems overall performance. The first subsystem is Municipality Supply line and it supplies water to rest three subsystems. On failure of supply of water from Municipality Supply line, the system does not have any water so whole system fails.

Key Words : *Supplementary variables, Laplace transform, Steady-state behaviour, Availability and cost function.*

The second subsystem is Tank and it interacts with human user. It stores the water supplied by the municipality. On failure of Tank the whole system gets failed (for example Leakage). The third subsystem is water purifier and it purifies the water and make it bacteria free and pure. On failure of water purifier, the whole system goes to failed state. The fourth subsystem is tap and it receives pure water from water purifier. In this model, there are two taps working in parallel redundancy. Therefore, on failure of any one tap, the whole system works in reduced efficiency state. Head-of-line policy has been adopted for repair purpose.

Since the considered system is of Non-Markovian nature [3], [4], supplementary variables have been used to make it Markovian. State-transition diagram has been shown in fig-1. Probability considerations and limiting procedure have been used for mathematical formulation of the system. This mathematical model has been solved with the aid of Laplace transform.

2. Assumptions

The following assumptions have been associated with this model:

1. Initially, the whole system is good and operable.
2. All failures follow exponential time distribution and are S-independent.
3. All repairs follow general time distribution and are perfect.
4. Head-of-line policy has been adopted for repair purpose.
5. There are two taps working in parallel redundancy.
6. On failure of any one tap, the whole system works in degraded state.
7. Repair facilities are always available and there is no time lap between a failure and start of repair.

3. Nomenclature

a, b, c, g	Failure rates of Tank, tap, Water purifier and Municipality Supply line, respectively.
$\mu_i(j)(t)$	The first order probability that i-th failure can be repaired in the time interval $(j, j+\Delta)$ conditioned that it was not repaired up to the time j.
$P_{0,0,0}(f)$	Pr {at time t, system is all operable}.
$P_{F,0,0}(x,t)\Delta$	Pr {at time t, system is failed due to failure of first subsystem A.T.M.}. Elapsed repair time lies in the interval (x, x, Δ).
$P_{-,F,0}(y,t)\Delta/P_G(m,t)\Delta$	Pr {at time t, system is failed due to failure of water purifier/ Municipality Supply line }. Elapsed repair time lies in the interval $y, y+\Delta)/(m, m+\Delta)$.
$P_{0,0,D}(z,t)\Delta$	Pr (at time t, system is degraded due to failure of any one tap}. Elapsed repair time lies in the interval $(z, z+\Delta)$.
$P^G_{0,0,D}(z,t)\Delta$	Pr (at time t, system is failed due to failure of Municipality Supply line while one tap has been failed already}. Elapsed repair time for tap lies in the interval $(z, z+\Delta)$.
$P_{F,0,D}(z,t)\Delta/P_{o,F,D}(z,t)\Delta$	Pr (at time t, system is failed due to failure of tank./ Water purifier while one tap has already been failed}. Elapsed repair time for tap lies in the interval (z, z, Δ).
$P_{0,0,F}(n,t)\Delta$	Pr (at time t, system is failed due to failure of any two taps}. Elapsed repair time lies in the interval $(n, +\Delta)$.
$S_i(t)$	$\mu_i(t)\exp\{-\int \mu_i(t)dt\}$
$D_i(s)$	$1-\overline{S}_i(s)/s$.

4. Formulation of Mathematical Model

Probability considerations and limiting procedure [1], [5] yield the following set of difference-differential equations, which is continuous in time and discrete in space, governing the behavior of considered system:

$$\left(\tfrac{d}{dt}+a+3b+c+g\right)P_{0,0,0}(t) = \int_0^\infty P_{F,0,0}()x,t)\mu_A(x)dx+\int_0^\infty P_{0,F,0}(y,t)\mu_C(y)dy + \int_0^\infty P_G(m,t)\mu_G(m)dm+\int_0^\infty P_{0,0,D}(z,t)\mu_B(z)dz \tag{1}$$

$$\left(\frac{\partial}{\partial x}+\frac{\partial}{\partial t}+\mu_A(x)\right)P_{F,0,0}(x,,t)=0 \tag{2}$$

$$\left(\frac{\partial}{\partial y}+\frac{\partial}{\partial t}+\mu_C(y)\right)P_{0,F,0}(y,t)=0 \tag{3}$$

$$\left(\frac{\partial}{\partial n}+\frac{\partial}{\partial t}+\mu_G(m)\right)P_G(m,t)=0 \tag{4}$$

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}++a+2b+c+g+\mu_B(z)\right)P_{0,0,D}(z,t)=0 \tag{5}$$

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}+\mu_B(z)\right)P_{F,0,D}(z,t)=a\ P_{0,0,D}(z,t) \tag{6}$$

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}+\mu_B(z)\right)P_{0,F,D}(z,t)=c\ P_{0,0,D}(z,t) \tag{7}$$

$$\left(\frac{\partial}{\partial z}+\frac{\partial}{\partial t}+\mu_B(z)\right)P^G_{0,0,D}(z,t)=g\ P_{0,0,D}(z,t) \tag{8}$$

$$\left(\frac{\partial}{\partial n}+\frac{\partial}{\partial t}+\mu_2(n)\right)P_{0,0,F}(n,t)=0 \tag{9}$$

Boundary conditions are

$$P_{F,0,0}(0,t)=a\ P_{0,0,0}(t)+\int_0^\infty P_{F,0,D}(z,t)\mu_B(z)dz \tag{10}$$

$$P_{0,F,0}(0,t)=c\ P_{0,0,0}(t)+\int_0^\infty P_{0,F,D}(z,t)\mu_B(z)dz \tag{11}$$

$$P_G(0,t)=g\ P_{0,0,0}(t)+\int_0^\infty P^G_{0,0,D}(z,t)\mu_B(z)dz \tag{12}$$

$$P_{0,0,0}(0,t)=3b\ P_{0,0,0}(t)+\int_0^\infty P_{0,0,F}(n,t)\mu_{B2}(n)dn \tag{13}$$

$$P_{F,0,D}(0,t)=0 \tag{14}$$

$$P_{0,F,D}(0,t)=0 \tag{15}$$

$$P^G_{0,0,D}(0,t)=0 \tag{16}$$

$$P_{0,0,F}(0,t)=2b\ P_{0,0,D}(t). \tag{17}$$

Initial conditions are:

$P_{0,0,0}(0)=1$, otherwise all state probabilities at $t=0$ are zero. (18)

5. Solution of the Model

Taking Laplace transforms of equations (1) through (17) subjected to initial conditions (18), and then on solving them one by one [3], we obtain the following Laplace transforms of various transition-state probabilities, depicted in fig-1:

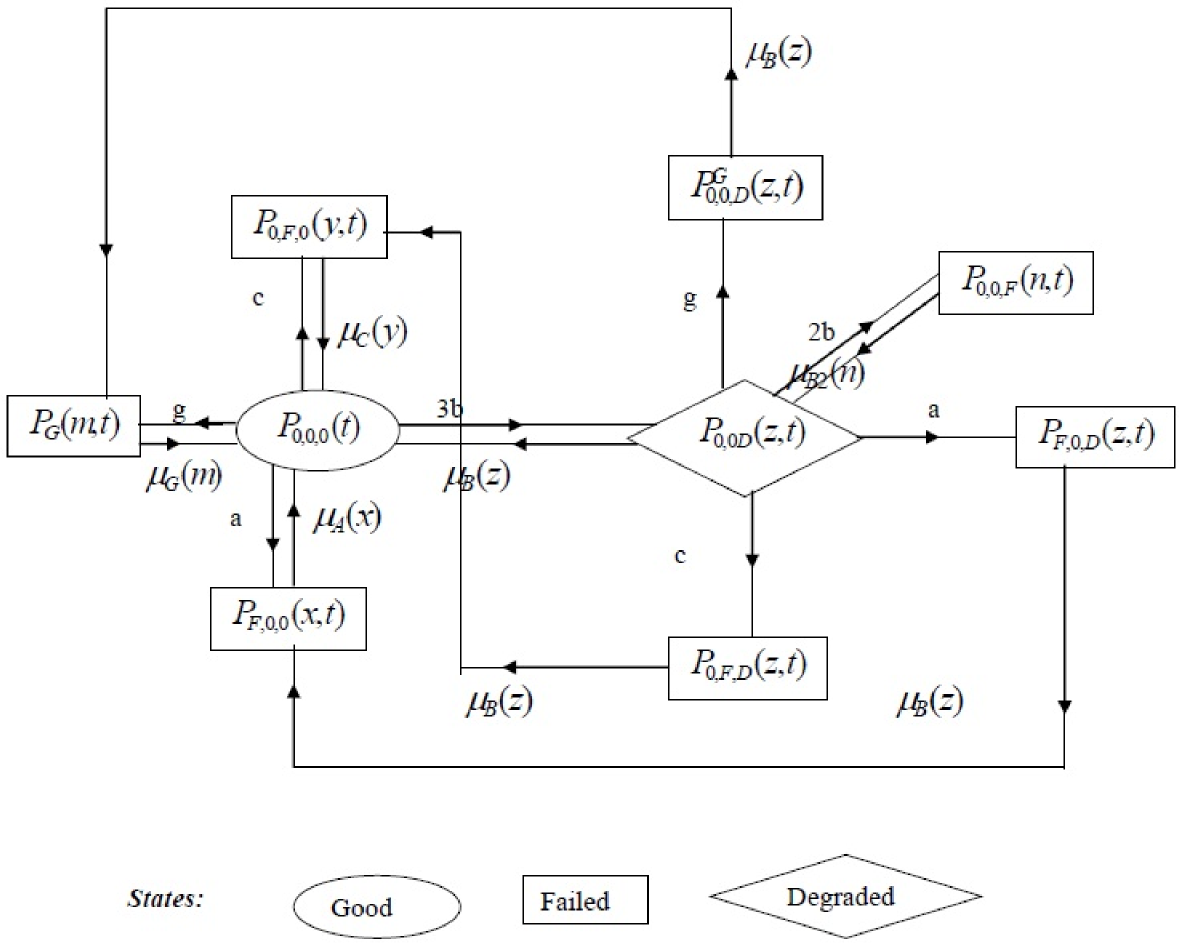


Fig- 1: State-transition diagram

$$\overline{P}_{0,0,0}(s) = \frac{1}{E(s)} \tag{19}$$

$$P_{F,0,D}(s) = \frac{aC(s)D_A(s)}{E(s)} \tag{20}$$

$$P_{0,F,0}(s) = \frac{cC(s)D_C(s)}{E(s)} \tag{21}$$

$$P_G(s) = \frac{gC(s)D_G(s)}{E(s)} \tag{22}$$

$$P_{0,0,D}(s) = \frac{A(s)}{E(s)} \tag{23}$$

$$P_{F,0,D}(s) = \frac{aB(s)}{E(s)} \tag{24}$$

$$P_{0,F,D}(s) = \frac{cB(s)}{E(s)} \tag{25}$$

$$P^{G}_{0,0,D}(s) = \frac{gB(s)}{E(s)} \tag{26}$$

$$P_{0,0,F}(s) = \frac{2b\ A(s)D_{N2}(s)}{E(s)} \tag{27}$$

where

$$A(s) = \frac{3b + D_B(s+a+2b+c+g)}{1 - 2bS_{B2}(s)D_B(s+a+2b+c+g)} \tag{28}$$

$$B(s) = \frac{3b + 2bA(s)\overline{S}_{B2}(s)}{a+2b+c+g}[D_B(s) - D_B(s+a+2b+c+g)] \tag{29}$$

and

$$\begin{aligned} E(s) \quad &= \quad s + a + 43b + c + g - aC(s)\overline{S}_A(s) - cC(s)\overline{S}_C(s) - gC(s)\overline{S}_G(s) \\ &\quad -\lfloor 3b + 2bA(s)\overline{S}_{B2}(s)\rfloor S_B(s+a+2b+c+g) \end{aligned} \tag{31}$$

It is interesting to note here that
Sum of equations (19) through (27) $= \frac{1}{s}$. (32)

6. Steady-state Behaviour of the System

By employing final value theorem on L.T., viz., $\lim\limits_{t\to\infty} P(t) = \lim\limits_{s\to 0} \overline{P}(s) = P$ (say), provided limit on left exits; to equations (19) through (27), we compute [2] the following steady-state behaviour of considered system:

$$P_{0,0,0} = \frac{1}{E(0)} \tag{33}$$

$$P_{F,0,0} = \frac{aC(0)M_A}{E(0)} \tag{34}$$

$$P_{0,F,0} = \frac{cC(0)M_C}{E(0)} \tag{35}$$

$$P_G = \frac{gC(0)M_G}{E(0)} \tag{36}$$

$$P_{0,0,D} = \frac{A(0)}{E(0)} \tag{37}$$

$$P_{F,0,D} = \frac{aB(0)}{E(0)} \tag{38}$$

$$P_{0,F,D} = \frac{cB(0)}{E(0)} \tag{39}$$

$$P^{G}_{0,0,D} = \frac{gB(0)}{E(0)} \tag{40}$$

$$P_{0,0,F} = \frac{2bA(0)M_{B2}}{E(0)} \tag{41}$$

where $M_i = -\overline{S}'_i(0)$ = Mean time to repair subsystem i $\;E(0) = \left\lfloor \frac{d}{ds}E(s) \right\rfloor_{s=0}$

$$A(0) = \frac{3bD_B(a + 2b + c + g)}{1 - 2bD_B(a + 2b + c + g)} \tag{42}$$

$$B(0) = \frac{3b + 2bA(0)}{a + 2b + c + g}[D_B(0) - D_B(a + 2b + c + g)] \tag{43}$$

$$C(0) = 1 + \frac{3b + 2bA(0)}{a + 2b + c + g}[1 - \overline{S}_B(a + 2b + c + g)] \tag{44}$$

7. Particular Case

When all repairs follow exponential time distribution

In this case, setting $\overline{S}_i(j) = \frac{\mu_i}{(j+\mu_i)}$, $\forall$ i and j, in equations (19) through (27), we obtained the following L. T. of various transition-states depicted in fig-1:

$$P_{0,0,0}(s) = \frac{1}{E_1(s)} \tag{45}$$

$$P_{F,0,0}(s) = \frac{aC_1(s)}{E_1(s)(s + \mu_A)} \tag{46}$$

$$P_{0,F,0}(s) = \frac{cC_1(s)}{E_1(s)(s + \mu_C)} \tag{47}$$

$$P_G(s) = \frac{gC_1(s)}{E_1(s)(s + \mu_G)} \tag{48}$$

$$P_{0,0,D}(s) = \frac{A_1(s)}{E_1(s)} \tag{49}$$

$$P_{F,0,D}(s) = \frac{aB_1(s)}{E_1(s)} \tag{50}$$

$$P_{0,F,D}(s) = \frac{cB_1(s)}{E_1(s)} \tag{51}$$

$$P^{G}_{0,0,D}(s) = \frac{gB_1(s)}{E_1(s)} \tag{52}$$

$$P_{0,0,F}(s) = \frac{2bA_1(s)}{E_1(s)(s+\mu_{B2})} \tag{53}$$

where

$$A(s) = \frac{3b(s+\mu_{B2})}{s^2 + s(a+2b+c+g+\mu_B+\mu_{B2}) + \mu_{B2}(a+c+g+\mu_B)} \tag{54}$$

$$B(s) = \frac{3b(s+\mu_{B2}) + 2bA_1(s)\mu_{B2}}{(s+\mu_B)(s+\mu_{B2})(s+a+2b+c+g+\mu_B)} \tag{55}$$

$$C_1(s) = 1 + B_1(s)\mu_B \tag{56}$$

and

$$\begin{aligned} E_1(s) \quad &= \quad s+a+3b+c+g-\tfrac{aC_1(s)\mu_A}{s+\mu_A}-\tfrac{cC_1(s)\mu_C}{s+\mu_C}-\tfrac{gC_1(s)\mu_G}{s+\mu_G} \\ &\quad -\left[3b+2bA_1(s)\tfrac{\mu_{B2}}{s+\mu_{B2}}\right]\tfrac{\mu_B}{s+a+2c+c+g+\mu_B} \end{aligned} \tag{57}$$

8. Availability of the System

Availability [3] of considered system is given by

$$P_{up}(s) = \frac{1}{s+A}\left[1+\frac{3b}{s+(A-b)}\right]$$

where $A = a+3b+c+g$.

Taking inverse Laplace transform, we get

$$P_{up}(t) = 3\exp\{-(A-b)t\} - 2\exp\{-At\} \tag{58}$$

Also

$$P_{down}(t) = 1 - P_{up}(t). \tag{59}$$

9. Cost Function for the Considered System

Cost function $G(t)$ is given by

$$G(t) = C_1\int_0^t P_{up}(t)dt - C_2t - C_3 \tag{60}$$

where C_1 and C_2 are revenue and repair costs per unit up time, respectively and C_3 is system establishment cost per cycle. Also,

$$\int_0^t P_{up}(t)dt = \frac{2(e^{-At}-1)}{A} - \frac{3(e^{-(A-b)t}-1)}{(A-b)} \tag{61}$$

where $A = a + 3b + c + g$.

10. Numerical Illustration

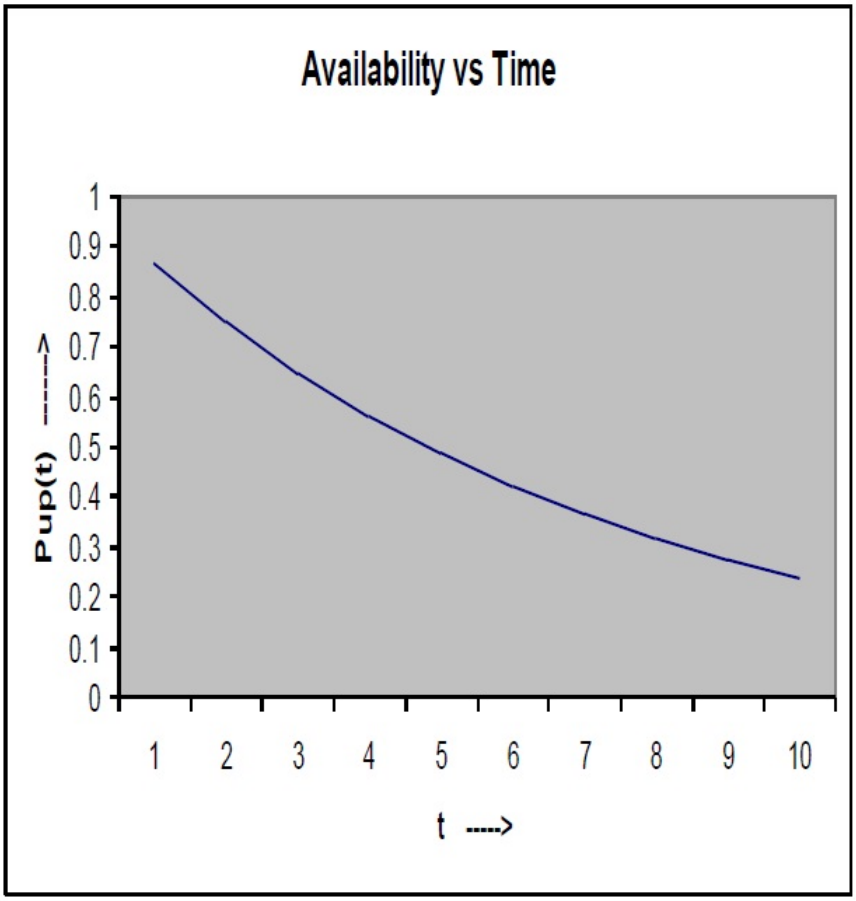
For a numerical illustration, let us consider the values:

$a = 0.08, b = 0.002, c = 0.004, g = 0.06, C_1 = Rs5.00, C_2 = Rs2.00, C_3 = Rs.10.00$ and $t = 0, 1, 2, \cdots, 10$.

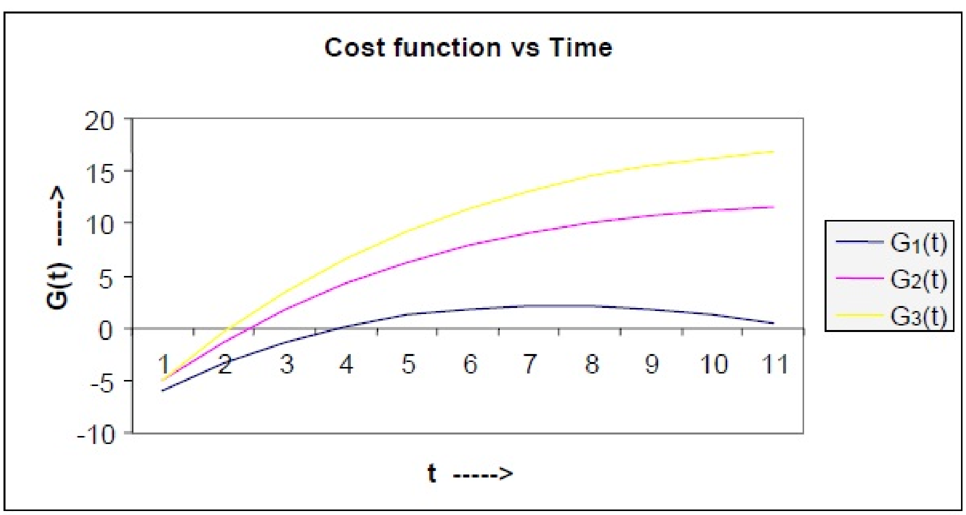
Using these values in equations (58) and (60), we have computed the tables (1) and (2), respectively. The corresponding graphs have been shown in fig-2 and 3, respectively.

11. Results and Discussion

We have given the availability of considered system, for various values of time t, in Table-1. Its graph has been shown in fig-2. Critical examination of fig-2 reveals that availability of the system decreases in a constant manner approximately. It should be noted that there are no sudden jumps in the values of availability of considered system. Table-2 gives the values of cost function at various t and for three sets of costs C_1, C_2, C_3. Its graph has been sketched in fig-3. In this graph, we observe that the cost function is negative in the beginning. This is because we have to invest a big amount C_3 to establish the system as a new. After that we recover this C_3 be C_1 slowly and value of cost function increases constantly. Also, we observe that value of G(t)remains better for the third set of C_1, C_2, C_3 (i.e. $G_3(t)$.



Availability vs Time
Pup(t) ------->
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1
2
3
4
5
6
7
8
9
10
t ----->



Cost function vs Time
G(t) ----->
20
15
10
5
0
-5
-10
1
2
3
4
5
6
7
8
9
10
11
G1(t)
G2(t)
G3(t)
t ----->

Table-1

$\mathbf{t}$	$\mathbf{P_{up}(t)}$
0	1
1	0.865877
2	0.749726
3	0.64914
4	0.562036
5	0.486609
6	0.421294
7	0.364738
8	0.315768
9	0.273366
10	0.236653

Table 2

$\mathbf{t}$	$\mathbf{G(t)}$		
	$G_1(t)$ $\mathbf{C_1 = 5}, C_2 = 2, C_3 - 6$	$G_2(t)$ $C_1 = 5, C_2 = 1, C_3 = 5$	$G_3(t)$ $C_1 = 6, C_2 = 1, C_3 = 5$
0	-6	-5	-5
1	-3.34334	-1.34334	-0.41201
2	-1.31129	1.688705	3.426446
3	0.179841	4.179841	6.615809
4	1.202559	6.202559	9.246071
5	1.819648	7.819648	11.38358
6	2.085489	9.085489	13.10259
7	2.047179	10.04718	14.45661
8	1.745507	10.74551	15.49461
9	1.215798	11.2158	16.25896
10	0.488641	11.48864	16.78637

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