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SOME FIXED POINT THEOREMS OF COMPATIBLE MAPPINGS OF TYPE (R) IN FUZZY METRIC SPACES

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Abstract

In this paper we proved some common fixed point theorems for four mappings of compatible Type (R) in fuzzy metric spaces.

Introduction and Preliminaries

The concept of Fuzzy sets was introduced at initially by Zadeh [7] which laid the foundation of fuzzy mathematics. After that, it was developed by many authors and used in various ways. Our paper deal with the fuzzy metric space defined by Kramosil and Michalek [9] and modified by George and Veeramani [1]. They also obtained that every metric space induces a fuzzy metric spaces. Jungck [4] introduced the concept of compatible mappings in 1986 by generalizing commuting mappings. Pathak, Chang and

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Cho [5, 6] introduced the concept of compatible mappings of type (P). Further, Singh and Chauhan [2] introduced the concept of compatible mappings of Fuzzy metric space and prove the common fixed point theorems. Using concept of compatible maps of type (α) and compatible maps of type (β) in Fuzzy metric space as introduced by Cho [13, 14]. Y. Rohen and others [14] introduced the concept of compatible mappings of type (C) by combining the definitions of compatible and compatible mappings of type (P) and later on it is renamed as compatible mapping of type (R) [15]. Dr. M. Singh and R. Gangil [3] used concept of compatible maps of type (R) and proved fixed point theorem of compatible mappings of type (R) in fuzzy metric space.

1. Preliminaries and Notations

Definition 1.1 : A binary operation $* : [0,1]\times[0,1] \to [0,1]$ is called t-norm if $([0,1]*)$ is an abelian topological monoid with unit 1 such that $a*b \leq c*d$, whenever $a \leq c$ and $b \leq d$ for a, b, c and $d \in [0,1]$. ([9])

Definition 1.2 : The triplet $(X, M, *)$ is a fuzzy metric space if X is an arbitrary set, $*$ is continuous t-norm, M is fuzzy set in $X^2 \times [0,\infty]$ satisfying the following conditions:

$F1 : M(x, y, 0) = 0$

$F2 : M(x, y, t) = 1$ for all $t > 0$ iff, $x = y$

$F3 : M(x, y, t) = M(y, x, t) \neq 0$ for $t \neq 0$

$F4 : M(x, y, t) = M(y, z, s) \leq M(x, z, t+s)$ for all $x, y, z \in X$ and $s, t > 0$

$F5 : M(x, y, .) : [0,\infty) \to [0,1]$ is left continuous,

$F6 : \lim_{t\to\infty} M(x, y, t) = 1$ for all x, y in X.

Note that $M(x, y, t)$ can be considered as the degree of nearness between x and y with respect to t. We identify $x = y$ with $M(x, y, t) = 1$ for all $t > 0$. ([9])

Definition 1.3 : Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be convergent to a point $x \in X$, if $\lim_{n\to\infty} M(x_n, x, t) = 1$, $\forall\ t > 0$ [9].

Definition 1.4 : Let $(X, M, *)$ be a fuzzy metric space. A sequence $\{x_n\}$ in X is said to be Cauchy sequence in X if $\lim_{n\to\infty} M(x_n; x_{n+p}; t) = 1$, $\forall\ t > 0$ and $p > 0$. [9]

Definition 1.5 : A fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence in X is converges to a point in X. [9]

Definition 1.6 : Let S and T be mappings from a complete fuzzy metric space $(X, M, *)$ into itself. The mappings S and T are said to be compatible mappings if

$M(STx_n, TSx_n, t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n; Tx_n \to p$ for some p in X as $n \to \infty$. [9]

Definition 1.7 : Let S and T be mappings from a complete fuzzy metric space $(X, M, *)$ into itself. The mappings S and T are said to be compatible mappings of type (P) if $M(SSx_n; TTx_n; t) \to 1$ for all $t > 0$, whenever $f\{x_n\}$ is a sequence in X such that $Sx_n; Tx_n \to p$ for some p in X as $n \to \infty$. [9]

Definition 1.8 : Let S and T be mappings from a complete fuzzy metric space $(X, M, *)$ into itself. The mappings S and T are said to be compatible mappings of type (R) if $M(STx_n; TSx_n; t) \to 1$ and $M(SSx_n; TTx_n; t) \to 1$ for all $t > 0$, whenever $\{x_n\}$ is a sequence in X such that $Sx_n; Tx_n \to p$ for some p in X as $n \to \infty$. [9]

2. Main Section

We need the following Lemmas for our main result.

Lemma 2.1 [9] : Let S and T be mappings from a complete fuzzy metric space $(X, M, *)$ into itself. If a pair S, T is compatible of type (R) on X and $Sz = Tz$ for $z \in X$, then $STz = TSz = SSz = TTz$.

Lemma 2.2 [3] : Let S and T be mappings from a complete fuzzy metric space $(X, M, *)$ into itself. If a pair S, T is compatible of type (R) on X and if $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z$ for some $z \in$X, then we have

(i) $M(TSx_n, Sz, t) \to 1$ as $n \to \infty$ if S is continuous.

(ii) $M(STx_n, Tz, t) \to 1$ as $n \to \infty$ if T is continuous.

(iii) $M(TTx_n, Sz, t) \to 1$ as $n \to \infty$ if S is continuous.

(iv) $M(SSx_n, Tz, t) \to 1$ as $n \to \infty$ if T is continuous.

(v) $STz = TSz$ and $Sz = Tz$ if S and T are continuous at z.

Now we give our main theorem.

Theorem 2.3 : Let $(X, M, *)$ be a complete metric space and A, B, S and T be mappings from X into itself. Suppose that S and T are continuous mappings satisfying the following conditions:

$$A(X) \subset T(X) \text{ and } B(X) \subset S(X), \tag{1}$$

The pairs $\{A, S\}$ and $\{B, T\}$ are compatible of type (R), (2)

$$M(Ax, By, t) \leq \Phi\ (\max\{M(Sx, Ty, t), M(Sx, Ax, t), M(Ty, By, t),$$
$$\frac{1}{2}[M(Sx, By, t) + M(Ty, Ax, t)]\}) \tag{3}$$

for all $x, y \in X$, where $\Phi : [0, \infty) \to [0, \infty)$ is a non-decreasing and upper semi continuous function and $\Phi(t) < t$ for all $t > 0$. Then A, B, S and T have a unique common fixed point in X.

Proof : Since $A(X) \subset (X)$ and $B(X) \subset S(X)$, we can choose a sequence $\{x_n\}$ in X such that

$$Sx_{2n} = Bx_{2n-1} \text{ and } Tx_{2n-1} = Ax_{2n-2} \text{ for } n = 1, 2, 3, \cdots .$$

Suppose that

$$y_{2n-1} = Tx_{2n-1} = Ax_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Bx_{2n-1} \tag{4}$$

for $n = 1, 2, 3, \cdots$.

By conditions (3) and (4), we have

$$\begin{aligned}
M(y_{2n+1}, y_{2n}, t) &= M(Ax_{2n}, Bx_{2n-1}, t) \\
&\leq \Phi(\max\{M(Sx_{2n}, Tx_{2n-1}, t), M(Sx_{2n}, Ax_{2n}, t), M(Tx_{2n-1}, Bx_{2n-1}, t), \\
&\quad \frac{1}{2}[M(Sx_{2n}, Bx_{2n-1}, t) + M(Tx_{2n-1}, Ax_{2n}, t)]\}) \\
&= \Phi(\max\{M(y_{2n}, y_{2n-1}, t), M(y_{2n}, y_{2n+1}, t), M(y_{2n-1}, y_{2n}, t), \\
&\quad \frac{1}{2}[M(y_{2n}, y_{2n}, t) + M(y_{2n-1}, y_{2n+1}, t)]\}) \\
&= \Phi M(y_{2n}, y_{2n-1}, t) \\
&\leq M(y_{2n}, y2_{n-1}, t).
\end{aligned}$$

Now we show that $\lim_{n\to\infty} M(y_{n+p}, y_n, t) = 1$, for all p and upper semi continuity of $\Phi(t) < t$ for all $t > 0$.

$$\begin{aligned}
M(y_{n+1}, y_n, t) &\leq M(y_n, y_{n-1}, t) \\
&\leq M(y_{n-1}, y_{n-2}, t) \\
&\leq M(y_{n-2}, y_{n-3}, t) \\
&\quad \vdots \\
&\leq M(y_1, y_0, t) \to 1. \ As \ \ t > 0 \ \ as \ \ n \to \infty.
\end{aligned}$$

Hence $\{y_n\}$ is a Cauchy sequence in X and so, since X is complete, it converges to a point z in X. On the other hand, the subsequences $\{Ax_{2n-2}\}, \{Bx_{2n-1}\}, \{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ of $\{y_n\}$ also converge to the point z.

Since $\{A, S\}$ and $\{B, T\}$ are compatible of type (R), it follows from the continuity of S and T, (4) and Lemma 2.2 that

$$Ty_{2n} \to Tz, \; By_{2n} = BBx_{2n-1} \to Tz,$$
$$Sy_{2n-1} \to Sz, Ay_{2n-1} = AAx_{2n-2} \to Sz \tag{5}$$

as $n \to \infty$.

By (3) and (4), we have

$$\begin{aligned} M(Ay_{2n-1}, By_{2n}, t) \;\leq\; & \Phi(\max\{M(Sy_{2n-1}, Ty_{2n}, t), M(Sy_{2n-1}, Ay_{2n-1}, t), M(Ty_{2n}, By_{2n}, t), \\ & \frac{1}{2}[M(Sy_{2n-1}, By_{2n}, t) + M(Ty_{2n}, Ay_{2n-1}, t)]\}). \end{aligned}$$

By the upper semicontinuity of $\Phi(t)$, (4) and (5), if $Sz \neq Tz$, then we have

$$\begin{aligned} M(Sz, Tz, t) \;&\leq\; \Phi(\max\{M(Sz, Tz, t), 0, 0, M(Sz, Tz, t)\}) \\ &=\; \Phi(M(Sz, Tz, t)) < M(Sz, Tz, t), \end{aligned}$$

which is contradiction. Thus it follows that $Sz = Tz$.

Similarly, from (3), (4), (5) and the upper semicontinuity of Φ, we can obtain $Sz = Bz$ and $Tz = Az$. Hence we have

$$Az = Bz = Sz = Tz. \tag{6}$$

From (3) and (4), we have also

$$\begin{aligned} M(Ax_{2n}, Bz, t) \;\leq\; & \Phi(\max\{M(Sx_{2n}, Tz, t), M(Sx_{2n}, Ax_{2n}, t), M(Tz, Bz, t), \\ & \frac{1}{2}[M(Sx_{2n}, Bz, t) + M(Tz, Ax_{2n}, t)]\}). \end{aligned}$$

This implies that, if $Bz \neq z$, then

$$M(z, Bz, t) \leq \Phi(M(z, Bz, t)) < M(z, Bz, t),$$

which is a contradiction. Therefore, we have $z = Az = Bz = Sz = Tz$. The uniquencesss of the fixed point z is obvious from (2). This completes the proof.

From Theorem (2.3), we have the following:

Theorem 2.4 : Let $(X, M, *)$ be a complete metric space and A, B be mappings from X into itself satisfying the following condition

$$\begin{aligned} M(Ax, By, t) \leq\ & \Phi(\max\{M(x,y,t), M(x,Ax,t), M(y,By,t), \\ & \tfrac{1}{2}[M(x,By,t) + M(y,Ax,t)]\}) \end{aligned} \tag{7}$$

for all x, y in X, where $\Phi(t)$ is the same as in Theorem 2.3 then A and B have a unique common fixed point in X.

Proof : Define a sequence $\{x_n\}$ in X by

$$x_{2n-1} = Ax_{2n-2} \text{ and } x_{2n} = Bx_{2n-2} \tag{8}$$

for $n = 1, 2, 3, \cdots$. Then it is easy to show that $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, letting $x_n \to z \in X$ as $n \to \infty$ we know that $\{x_{2n-1}\}$ and $\{x_{2n}\}$ converge to z, too. By (7) and (8), we have

$$\begin{aligned} M(Az, x2n, t) &\leq M(Az, Bx_{2n-2}, t) \\ &\leq \Phi(\max\{M(z, x_{2n-2}, t), M(z, Az, t), M(x_{2n-2}, x_{2n}, t), \\ &\quad \frac{1}{2}[M(z, x_{2n}, t) + M(x_{2n-2}, Az, t)]\}). \end{aligned}$$

By the upper semicontinuity of $\Phi(t)$, if $Az \neq z$, then we have

$$M(Az, z, t) \leq \Phi(M(z, Az, t)) < M(z, Az, t),$$

which is contradiction and so $z = Az$. Similarly, we have $z = Bz$. This completes the proof.

The following result is an immediate consequence of Theorem 2.3.

Theorem 2.5 : Let $(X, M, *)$ be a complete metric space and S, T and A_n be mappings from X into itself, $n = 1, 2, \cdots$. Suppose further that S and T are continuous and, for every $n \in N$, the pairs $\{A_{2n-1}, S\}$ and $\{A_{2n}, T\}$ are compatible of type (R), $A_{2n-1}(X) \subset T(X)$ and $A_{2n}(X) \subset S(X)$ and for any $n \in N$, the set of positive integers, the following condition is satisfied:

$$\begin{aligned} M(A_n x, A_{n+1} y, t) \leq\ & \Phi(\max\{M(Sx, Ty, t), M(Sx, A_n x, t), M(Ty, A_{n+1} y, t), \\ & \tfrac{1}{2}[M(Sx, A_{n+1} y, t) + M(Ty, A_n x, t)]\}). \end{aligned} \tag{9}$$

for all $x, y \in X$, where $\Phi(t)$ is the same as in Theorem 2.3. Then S, T and $\{A_n\}$, $n \in N$, have a unique common fixed point in X.

3. Conclusion

In this paper, we established some fixed pont theorem for four self maps of compatible maps of type (R) which generalized some valuable results of Y. Rohen.

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