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$\mathbb{C}_2$-NETS AND $\mathbb{C}_0$(o)-TOPOLOGY

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Abstract

In this paper, we define the nets with bicomplex entries (called as $\mathbb{C}_2$-nets) and analyse the $\mathbb{C}_0$(o)-topology on $\mathbb{C}_2$. The space $\mathbb{C}_2$ equipped with this topology elaborates interesting and challenging behaviour of $\mathbb{C}_2$-nets. The higher dimension of space $\mathbb{C}_2$, play an important role in the various capabilities of $\mathbb{C}_2$-nets called confluences. Methods of confluence are constructed in a sense of convergence of the component nets of $\mathbb{C}_2$-nets.

1. Introduction

The well known extension of the complex numbers to the higher dimension is the quaternion by Hamilton. And it represents the rotation of the three-dimensional space. The set of quaternion is a non-commutative division algebra. One more extension is known as bicomplex numbers. The concept of bicomplex numbers was given by Segre [6] in

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1892. Almost after 100 years, the emeritus professor G.B. Price in Italy had published a monograph [3] on multicomplex spaces and functions, where he had provided the details about the bicomplex algebra. Later on, Rochon and Shapiro [14] had given the details about algebraic properties, conjugation and moduli of the bicomplex numbers and Shapiro *et al.* [4] and Ronn [15] discussed the bicomplex functions and their basic properties.

It was further investigated from the functional analysis point of view and linked with the spectral theory and bicomplex topological modules by Struppa *et al.* [5, 4], Rajeev *et al.* Charak *et al.* [1] and on holomorphic functional calculus by Struppa [5], on Hilbert spaces by Rochon [13] and H. M. Campos et V. V. Kravchenko has studied fundamentals of bicomplex pseudo-analytic function theory in [2].

Both extensions of the three dimensional space are quit different because the quaternion form a non-commutative division ring but the set of all bicomplex numbers is a commutative ring but not a division ring. In this paper, our work is focused on the bicomplex space. Throughout the paper, the set of real numbers, complex numbers and bicomplex numbers are denoted by $\mathbb{C}_0$, $\mathbb{C}_1$ and $\mathbb{C}_2$, respectively.

The set of bicomplex numbers is defined as follows:

$$\mathbb{C}_2 := \{\xi \,:\, \xi = x_1 + i_1x_2 + i_2x_3 + jx_4;\ x_p \in \mathbb{C}_0, 1 \leq p \leq 4\}$$

or equivalently

$$\mathbb{C}_2 := \{\xi \,:\, \xi = z_1 + i_2z_2; z_1, z_2 \in \mathbb{C}_1\}$$

where i_1 and i_2 are commuting imaginary units with the properties:

$$i_1^2 = i_2^2 = -1 \quad \text{and} \quad i_1i_2 = i_2i_1 = j$$

We shall use the notation $\mathbb{C}(i_p)$ $(p = 1, 2)$ for the following set:

$$\mathbb{C}(i_p) := \{x_1 + i_px_2 \,:\, i_p^2 = -1\,\text{and}\ x_1,\, x_2 \in \mathbb{C}_0\}$$

We observe that the set $\mathbb{C}_1$ is homeomorphic to the sets $\mathbb{C}(i_1)$ and $\mathbb{C}(i_2)$ the usual topology on $\mathbb{C}_1$. The auxiliary spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ are defined as follows [8]:

$$\mathbb{A}_1 := \{w_1 - i_1w_2 \,:\, w_1, w_2 \in \mathbb{C}_1\} = \left\{{}^1\zeta \,:\, \zeta \in \mathbb{C}_2\right\}$$
$$\mathbb{A}_2 := \{w_1 + i_1w_2 \,:\, w_1, w_2 \in \mathbb{C}_1\} = \left\{{}^2\zeta \,:\, \zeta \in \mathbb{C}_2\right\}$$

Both subsidiary spaces $\mathbb{A}_1$ and $\mathbb{A}_2$ are homeomorphic to $\mathbb{C}(i_p)$, $p = 1, 2$ as well as $\mathbb{C}_1$. The bicomplex number ξ can also be written as:

$$\xi = z_1 + i_2 z_2 = {}^1\zeta e_1 + {}^2\zeta e_2$$

where

$$^1\zeta = z_1 - i_1 z_2 \in \mathbb{A}_1 \quad \text{and} \quad ^2\zeta = z_1 + i_1 z_2 \in \mathbb{A}_2$$

and the bicomplex numbers e_1 and e_2 are defined as

$$e_1 = \frac{1+j}{2} \quad \text{and} \quad e_2 = \frac{1-j}{2}$$

The bicomplex numbers e_1 and e_2 are hyperbolic numbers and

$$e_1^2 = e_1, \quad e_2^2 = e_2, \quad e_1 + e_2 = 1 \quad \text{and} \quad e_1 e_2 = 0$$

From the above equations, we can say that e_1 and e_2 are non-trivial idempotent elements as well as the non-trivial zero divisors in the bicomplex space. Hence, bicomplex space $\mathbb{C}_2$ is not an integral domain. The existence of the non-trivial zero divisors is one of the major difference between the algebraic structures of the complex and bicomplex spaces. A norm on the bicomplex linear space is given below: [3] :

$$||\eta|| = \sqrt{a_1^2 + a_2^2 + a_3^2 + a_4^2} = \sqrt{|z_1|^2 + |z_2|^2} = \sqrt{\frac{|^1\eta|^2 + |^2\eta|^2}{2}} \tag{1.1}$$

With this norm, $\mathbb{C}_2$ becomes a commutative Banach algebra. Further,

$$||\eta \times \zeta|| \leq \sqrt{2}\ ||\eta|| \,.\, ||\zeta||\,, \quad \forall\, \eta, \zeta \in \mathbb{C}_2$$

It is the best possible inequality. Because of this deficiency, $\mathbb{C}_2$ is known as *modified complex Banach algebra* [3].

Remark 1.1 : For the idempotent elements, we have:

$$||e_p \cdot e_p|| = \sqrt{2}\, ||e_p||\, ||e_p||, \qquad p = 1, 2.$$

We enlarged the study of topological structures of the space $\mathbb{C}_2$ which was initiated by Srivastava [8]. He characterized some topologies in the usual sense by using different properties of the bicomplex space. We continue the topological study of $\mathbb{C}_2$ by the lexicographical order relation.

2. Definitions and Preliminaries

There are a lot of results which are true in nets but not for sequences. The nets are giving many advantages over sequences. On the bicomplex space $\mathbb{C}_2$, we have defined a lexicographic order relation namely $\ell_{\mathbb{C}_0}$-order and defined an order topology, called $\mathbb{C}_0$(o)-topology on $\mathbb{C}_2$. We also tried to develop the ideal of nets on the bicomplex space. We studied the confluences (sometimes called convergence in some sense) of the $\mathbb{C}_2$-nets with respect to the $\mathbb{C}_0$(o)-topology. We improved many results, definition and examples given by us on the topologies on $\mathbb{C}_2$. Throughout in this section, η and ζ will denote bicomplex numbers defined by $\eta = a_1 + i_1a_2 + i_2a_3 + ja_4$ and $\zeta = b_1 + i_1b_2 + i_2b_3 + jb_4$.

Definition 2.1 : [**$\ell_{\mathbb{C}_0}$-order**] For any two bicomplex numbers η and ζ, we have the relation, $\eta \prec_{\mathbb{C}_0} \zeta$, if $a_p \leq b_p$, for some $p \in \mathbb{N}$, $1 \leq p \leq 4$ and $a_q = b_q$, for $q \in \mathbb{N}$, $1 \leq q < p$.

The reverse order of $\ell_{\mathbb{C}_0}$-order is also a linear order. The $\ell_{\mathbb{C}_0}$-order is a linear order and the notation of open intervals $(\xi, \eta)_{\mathbb{C}_0}$ denote a basis element of an order topology on $\mathbb{C}_2$ w.r.t the $\ell_{\mathbb{C}_0}$-order. Now, we have

$$\begin{aligned}
(\xi, \rightarrow)_{\mathbb{C}_0} &= \{\zeta \in \mathbb{C}_2 : \xi \prec_{\mathbb{C}_0} \zeta\} \\
(\leftarrow, \eta)_{\mathbb{C}_0} &= \{\zeta \in \mathbb{C}_2 : \zeta \prec_{\mathbb{C}_0} \eta\} \\
(\xi, \eta)_{\mathbb{C}_0} &= \{\zeta \in \mathbb{C}_2 : \xi \prec_{\mathbb{C}_0} \zeta \prec_{\mathbb{C}_0} \eta\}.
\end{aligned}$$

In the similar manner, the intervals $[\xi, \rightarrow)_{\mathbb{C}_0}$, $(\leftarrow, \eta]_{\mathbb{C}_0}$, $(\xi, \eta)_{\mathbb{C}_0}]$ and $[\xi, \eta)_{\mathbb{C}_0}$ can be defined.

Definition 2.2 : Consider that $\xi \prec_{\mathbb{C}_0} \eta$. Then the following four collections of various forms of open intervals are defined as:

1. $G_1 = \{(a_1 + i_1a_2 + i_2a_3 + ja_4,\, b_1 + i_1b_2 + i_2b_3 + jb_4)_{\mathbb{C}_0} : a_1 < b_1\}$
2. $G_2 = \{(a_1 + i_1a_2 + i_2a_3 + ja_4,\, a_1 + i_1b_2 + i_2b_3 + jb_4)_{\mathbb{C}_0} : a_2 < b_2\}$
3. $G_3 = \{(a_1 + i_1a_2 + i_2a_3 + ja_4,\, a_1 + i_1a_2 + i_2b_3 + jb_4)_{\mathbb{C}_0} : a_3 < b_3\}$
4. $G_4 = \{(a_1 + i_1a_2 + i_2a_3 + ja_4,\, a_1 + i_1a_2 + i_2a_3 + jb_4)_{\mathbb{C}_0} : a_4 < b_4\}$.

Suppose $\mathbb{B} = \bigcup_{i=1}^{4} G_i$. The sets of the type $[x_1 = a]_{\mathbb{C}_0}$, $[x_1 = a, x_2 = b]_{\mathbb{C}_0}$, $[x_1 = a, x_2 = b, x_3 = c]_{\mathbb{C}_0}$ are known as R-frame, R-plane and R-line and members of G_1, G_2, G_3 and G_4 are called as R-space segments, R-frame segments, R-plane segments, and R-line segments (or R-intervals) in the $\ell(\mathbb{C}_0)$-order relation.

Lemma 2.3 [11] : The family $\mathbb{B}$ is a basis for some topology on the bicomplex space.

Remark 2.4 : We shall call the topology generated by $\mathbb{B}$ as $\mathbb{C}_0$*(o)-topology* and it is denoted as τ_4. Also the collection $\{(\xi, \rightarrow)_{\mathbb{C}_0} : \xi \in \mathbb{C}_2\} \cup \{(\leftarrow, \xi)_{\mathbb{C}_0} : \xi \in \mathbb{C}_2\}$ forms a subbasis for this topology. The interval $(\xi, \eta)_{\mathbb{C}_0} = (a_1 + i_1a_2 + i_2a_3 + ja_4,\ b_1 + i_1b_2 + i_2b_3 + jb_4)_{\mathbb{C}_0}$ called as $\mathbb{C}_0$-open set in τ_4. It may be R-frame segment, R-plane segment, R-line segment and R-interval depends on the order of elements of the interval.

Here, we constructed the $\mathbb{C}_2$-nets and also defined the various types of confluences of these nets w.r.t. the $\mathbb{C}_0$(o)-topology. We also defined the $\mathbb{C}_2$-subnets of the $\mathbb{C}_2$-nets and studied them.

Definition 2.5 (Directed Set) : Let D be a partially ordered set with the order relation $\geq$, then D is called as a *directed set* if for any two elements $a, b \in D$, $\exists$ some $c \in D$ such that $c \geq a$ and $c \geq b$.

Every totally ordered set is a directed set. The product of two directed sets is a directed set. Further, the subset of a directed set is a directed set.

Definition 2.6 ($\mathbb{C}_2$-Net) [12] : For some directed set D, the $\mathbb{C}_2$-net can be defined as $f : D \rightarrow \mathbb{C}_2$ such that $\forall\, \alpha \in D$

$$\begin{aligned} f(\alpha) &= y_{1\alpha} + i_1y_{2\alpha} + i_2y_{3\alpha} + jy_{4\alpha} \\ &= w_{1\alpha} + i_2w_{2\alpha} \\ &= {}^1\eta_\alpha e_1 + {}^2\eta_\alpha e_2. \end{aligned} \tag{2.2}$$

We denote the $\mathbb{C}_2$-net $f(\alpha)$ as $\{\eta_\alpha\}_{\alpha \in D}$ or $\{\eta_\alpha\}$, and D will denote the directed set. Also, a tail T_α in the directed set $(D, \geq)$ is the set $T_\alpha = \{\beta : \beta \geq \alpha\}$.

Definition 2.7 ($\mathbb{C}_0$(F)-Confluence) : The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is said to be $\mathbb{C}_0$*(F)-confluence* to the R-frame $[x_1 = a]_{\mathbb{C}_0}$, if for every $\beta \in D$, there exists $N \in G_1$ such that $\eta_\alpha \in N$, $\forall \alpha \geq \beta$ and $[x_1 = a]_{\mathbb{C}_0} \subset N$.

Definition 2.8 ($\mathbb{C}_0$(P)-Confluence) : The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is said to be $\mathbb{C}_0$*(P)-confluence* to the R-plane $[x_1 = a, x_2 = b]_{\mathbb{C}_0}$, if for every $\beta \in D$, there exists $N \in G_2$ such that $\eta_\alpha \in N$, $\forall \alpha \geq \beta$ and $[x_1 = a, x_2 = b]_{\mathbb{C}_0} \subset N$.

Definition 2.9 ($\mathbb{C}_0$(L)-Confluence) : The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is said to be $\mathbb{C}_0$*(L)-confluence* to the R-line $[x_1 = a, x_2 = b, x_3 = c]_{\mathbb{C}_0}$, if for every $\beta \in D$, there exists $N \in G_3$ such that $\eta_\alpha \in N$, $\forall \alpha \geq \beta$ and $[x_1 = a, x_2 = b, x_3 = c]_{\mathbb{C}_0} \subset N$.

Definition 2.10 ($\mathbb{C}_0$-Point Confluence): The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is said to be $\mathbb{C}_2$-*Point confluence* to the $\eta = a + i_1 b + i_2 c + jd$, if for every $\beta \in D$, there exists $N \in G_4$ such that $\xi_\alpha \in N$, $\forall \alpha \geq \beta$ and $\eta \in N$. It is denoted as $\mathbb{C}_0$-$\lim\limits_{\alpha \in D} \eta_\alpha = \eta$.

Remark 2.11 : A $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is *stable* on η if $\eta_\alpha = \eta$, $\forall\, \alpha \in D$. It is *finally stable* on η if there exists $\beta \in D$ such that $\eta_\alpha = \eta$, $\forall\, \alpha \geq \beta$. Throughout this paper, we shall emphasise to $\mathbb{C}_2$-net in the form

$$\{\eta_\alpha\} = \{y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha}\},\ y_{k\alpha} \in \mathbb{C}_0, \forall \alpha \in D \text{ and } k = 1, 2, 3, 4.$$

Remark 2.12 : A $\mathbb{C}_2$-net can be finally stable at a point but not at any R-frame, R-plane or R-line w.r.t. $\mathbb{C}_2$(o)-topology. Further, If the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(F)-confluence to an R-frame $[x_1 = a]_{\mathbb{C}_0}$, then it may not finally in any member of the collection G_2 (and hence, it cannot be $\mathbb{C}_0$(P)-confluence to any R-plane) unless $\{y_{1\alpha}\}$ is finally stable on a, say (and in case, $\{\eta_\alpha\}$ will be $\mathbb{C}_0$(P)-confluence to the R-plane $[x_1 = a,\ x_2 = b]_{\mathbb{C}_0}$ provided that $\{y_{2\alpha}\}$ converges to b). Analogous observations can be derived with the other forms of $\mathbb{C}_2$-nets in real form.

Remark 2.13 : The $\mathbb{C}_0$-Point confluence of a $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is a necessary but not a sufficient condition for the net in the topology τ_1 defined in [8] induced by the Equation (1.1). In fact, every finally stable net $\{y_{k\alpha}\}$, $1 \leq k \leq 4$, converges and therefore, $\mathbb{C}_0$-Point confluence of $\{\eta_\alpha\}$ to ξ implies convergence of $\{\eta_\alpha\}$ to ξ in τ_1. To test this weakness, consider the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ defined on the directed set $(\mathbb{Q}^+, \geq)$ of positive rational (with usual order) as follows:

$$\eta_\alpha = y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha}, \forall \alpha \in D$$

where $y_{k\alpha} = 1 + \dfrac{1}{\alpha^2 + k^2}$, $1 \leq k \leq 4$.

Then this $\mathbb{C}_2$-net converges to a point $\xi = 1 + i_1 + i_2 + j$ in τ_1 but not $\mathbb{C}_0$-Point confluence to ξ in τ_4. But for the given R-line segment

$$(1 + i_1 + i_2 + (1 - \epsilon)j,\ 1 + i_1 + i_2 + (1 + \epsilon)j)_{\mathbb{C}_0}$$

no element of $\{\eta_\alpha\}$ is contained in the R-line segment.

3. Main Results

Some results on the $\mathbb{C}_0$-confluences of the $\mathbb{C}_2$-nets have been discussed in the section and the results of $\mathbb{C}_2$-subnet and Cauchy $\mathbb{C}_2$-net have been developed and studied.[1]

Theorem 3.1 : A $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(F)-confluence to the R-frame $[x_1 = k]_{\mathbb{C}_0}$ iff the net $\{y_{1\alpha}\}$ converges to k.

Proof : Let $\{y_{1\alpha}\}$ be a net converging to k. For a given $\delta > 0$, consider a set as follows:

$$B_\delta = \{\eta \,:\, \eta = x_1 + i_1 x_2 + i_2 x_3 + j x_4 \,;\, k - \delta < x_1 < k + \delta\} \tag{3.1}$$

Obviously, $B_\delta \in G_1$ and $[x_1 = k]_{\mathbb{C}_0} \subset B_\delta$.

Since, $\{y_{1\alpha}\}$ is converges to k, then $\exists\, \gamma \in D$ such that

$\eta_{1\alpha} \in (k - \delta,\, k + \delta),\ \forall\, \alpha \geq \gamma$

$\Rightarrow\ k - \delta < \eta_{1\alpha} < k + \delta,\ \forall\, \alpha \geq \gamma$

$\Rightarrow\ k - \delta + i_1 x_2 + i_2 x_3 + j x_4 \prec_{\mathbb{C}_0} y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha}$

and $y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha} \prec_{\mathbb{C}_0} k + \delta + i_1 y_2 + i_2 y_3 + j y_4$,

where $\forall\, x_r,\, y_r \in \mathbb{C}_0,\ 2 \leq r \leq 4$. Therefore,

$x_{1\alpha} + i_1 x_{2\alpha} + i_2 x_{3\alpha} + j x_{4\alpha} \in (k - \delta + i_1 x_2 + i_2 x_3 + j x_4,\, k + \delta + i_1 y_2 + i_2 y_3 + y_4)_{\mathbb{C}_0}$, $\forall\, x_r,\, y_r \in \mathbb{C}_0,\ 2 \leq r \leq 4$ and $\alpha \geq \gamma$.

So that the net $\{\eta_\alpha\} = \{y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha}\}$ is finally in B_δ.

Now, for every $N \in G_1$, there exists a B_δ, $(\delta > 0)$ such that $B_\delta \subset N$. Hence, $\{\eta_\alpha\}$ is $\mathbb{C}_0$(F)-confluenced to the R-frame $[x_1 = k]_{\mathbb{C}_0}$.

Conversely, let the $\mathbb{C}_2$-net $\{\eta_\alpha\} = \{y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha}\}$ be $\mathbb{C}_0$(F)-conflenced to R-frame $[x_1 = k]_{\mathbb{C}_0}$.

So it is finally in every member of G_1 containing the R-frame $[x_1 = k]_{\mathbb{C}_0}$.

Particularly, this $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is finally in B_δ $(\delta > 0)$ given in Equation (3.1).

Then, $\exists$ some β in D such that $\eta_\alpha \in B_\delta\,,\ \forall\, \alpha \geq \beta$

$\Rightarrow\ k - \delta < y_{1\alpha} < k + \delta,\ \ \forall\, \alpha \geq \beta.$

$\Rightarrow\ y_{1\alpha} \to k.$ □

Theorem 3.2 : The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(P)-confluence to $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0}$ iff the nets $\{y_{1\alpha}\}$ is finally stable on k and $\{y_{2\alpha}\}$ converges to ℓ.

Proof : Suppose that the $\mathbb{C}_2$-net $\{y_{1\alpha}\}$ is finally stable on k and $\{y_{2\alpha}\}$ converge to ℓ. As $\{y_{1\alpha}\}$ is finally stable on k, then there exists a $\gamma \in D$ such that $y_{1\alpha} = k$, for all

[1]The preliminaries of these results were presented in the 81st annual conference of the Indian Mathematical Society in VNIT Nagpur (India) in December 2015.

$\alpha \geq \gamma$. For a given $\epsilon > 0$, construct a set as follows:

$$U_\epsilon = \{\eta : \eta = k + i_1 a_2 + i_2 a_3 + j a_4 \,;\, \ell - \epsilon < a_2 < \ell + \epsilon\} \tag{3.2}$$

Then, $U_\epsilon \in G_2$ and $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0} \subset U_\epsilon$.

Now, for the coefficient net $\{y_{2\alpha}\}$ of the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ converges to ℓ, then there exists some $\gamma \in D$ such that

$$y_{2\alpha} \in (\ell - \epsilon,\, \ell + \epsilon),\ \forall\, \alpha \geq \gamma$$

Since for $\beta,\, \gamma \in D$, there exists some $\lambda \in D$ such that $\lambda \geq \beta$ and $\lambda \geq \gamma$.

Therefore, $y_{1\alpha} = k$ and $y_{2\alpha} \in (\ell - \epsilon,\, \ell + \epsilon),\ \forall\, \alpha \geq \lambda$.

$\Rightarrow\ \ell - \epsilon < y_{2\alpha} < \ell + \epsilon,\quad \forall\, \alpha \geq \lambda$.

$\Rightarrow\ y_{1\alpha} + i_1 y_{2\alpha} + i_2 y_{3\alpha} + j y_{4\alpha} \in (k + i_1(\ell - \epsilon) + i_2 a_3 + j a_4,\ k + i_1(\ell + \epsilon) + i_2 b_3 + j b_4)_{\mathbb{C}_0}$, $\forall\ a_3, a_4, b_3, b_4 \in \mathbb{C}_0,\ \forall\, \alpha \geq \lambda$.

Therefore, the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is finally in the set U_ϵ.

Now, for every $M \in G_2$, there exists a set U_ϵ, $(\epsilon > 0)$ such that $U_\epsilon \subset M$.

So that the $\mathbb{C}_0$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(P)-confluenced to the R-plane $[x_1 = k, x_2 = \ell]_{\mathbb{C}_0}$. This implies that for given $\epsilon > 0$, there exists $\gamma \in D$ such that $\eta_\beta \in U_\epsilon$, $\forall \beta \geq \gamma$.

Conversely, let the $\mathbb{C}_0$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(P)-confined to the R-plane $[x_1 = k,\, x_2 = \ell]_{\mathbb{C}_0}$. Therefore, it is finally in every U_ϵ, $(\epsilon > 0)$ as given in the Equation (3.2).

So, $y_{1\alpha} = k$ and $y_{2\alpha} \in (\ell - \epsilon,\, \ell + \epsilon),\quad \forall\, \alpha \geq \beta$.

Hence, $\{y_{1\alpha}\}$ is finally stable on k and the net $\{y_{2\alpha}\}$ is converging to ℓ. □

Analogously, some more theorems can be proved as given below. Proofs are omitted.

Theorem 3.3 : The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(L)-confluence to $[x_1 = k, x_2 = \ell, x_3 = m]_{\mathbb{C}_0}$ iff the nets $\{y_{1\alpha}\}$ and $\{y_{2\alpha}\}$ are finally stable on k and ℓ, respectively and the net $\{y_{3\alpha}\}$ is converging to m.

Theorem 3.4 : For a $\mathbb{C}_2$-net $\{\eta_\alpha\}$, $\mathbb{C}_0\text{-}\lim\limits_{\alpha \in D} \eta_\alpha = a + i_1 b + i_2 b + j d$ iff the nets $\{y_{1\alpha}\}$, $\{y_{2\alpha}\}$ and $\{y_{3\alpha}\}$ are finally stable on a, b and c, respectively and the net $\{y_{4\alpha}\}$ is converging to d.

Definition 3.5 : [**$\mathbb{C}_2$-Subnet**] The given $\mathbb{C}_2$-net $\{\eta_\beta\}_{\beta \in E}$ is said to be a $\mathbb{C}_2$-subnet of the $\mathbb{C}_2$-net $\{\xi_\alpha\}_{\alpha \in D}$ if for each tail T_α in D, there is a tail T_β in E such that $\{\eta_\delta : \delta \in T_\beta\} \subset \{\xi_\gamma : \gamma \in T_\alpha\}$.

Definition 3.6 (Cauchy $\mathbb{C}_2$-Net) : A $\mathbb{C}_2$-net $\{\xi_\alpha\}_D$ is said to be a Cauchy $\mathbb{C}_2$-net if the $\mathbb{C}_2$-net $\{\xi_\alpha - \xi_\beta\}_{D \times D}$ is $\mathbb{C}_0$-point cofluenced to zero.

Some results related to $\mathbb{C}_2$-subnets and Cauchy $\mathbb{C}_2$-net are proved as follows:

Theorem 3.7 : Every cofinal $\mathbb{C}_2$-subnet of a $\mathbb{C}_0$-point confluenced $\mathbb{C}_2$-net is also $\mathbb{C}_0$-point confluenced.

Proof : Let $\mathbb{C}_2$-net $\{\eta_\alpha\}_D$ be $\mathbb{C}_0$-point confluenced to the point η.

Now, if D' is a cofinal subset of the directed set D, then for every $\alpha \in D$, there exists a $\delta \in D'$ such that $\delta \geq \alpha$.

Also, D' as a subset of D, is a directed set under the same order relation of D.

Consider a $\mathbb{C}_2$-net $\{\eta_\lambda\}$ on the directed set D'.

Clearly, $\{\eta_\lambda\}$ is a cofinal $\mathbb{C}_2$-subnet of $\{\xi_\alpha\}$.

Then for each S_α of D there is a S_λ of D' such that for each $\gamma \in T_\lambda$, there is some $\delta \in T_\alpha$ such that

$$\{\eta_\lambda : \lambda \geq \gamma\} \subset \{\xi_\alpha : \alpha \geq \delta\}. \tag{3.3}$$

Now as the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is $\mathbb{C}_0$-point confluence to ξ, the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ is finally in an open R-line segments containing η.

From Equation (3.3), we have obtained that every tail of points of the $\mathbb{C}_2$-net $\{\xi_\alpha\}$ contains some tail of the points of the $\mathbb{C}_2$-subnet $\{\eta_\lambda\}$ and also D' is a cofinal subset of D.

Therefore, we conclude that the subnet $\{\eta_\lambda\}$ lies eventually in every member of the family N_4 containing the point ξ. Hence, $\{\eta_\lambda\}$ is $\mathbb{C}_0$-point confluence to ξ. □

Remark 3.8 : If domain of a $\mathbb{C}_2$-subnet of the given $\mathbb{C}_2$-net is not cofinal subset of domain of the $\mathbb{C}_2$-net, then the $\mathbb{C}_0$-confluence to an $\mathbb{C}_0$-confluence zone of the $\mathbb{C}_2$-net may or may not imply the $\mathbb{C}_0$-confluence of the $\mathbb{C}_2$-subnet to the same $\mathbb{C}_0$-confluence zone.

Remark 3.9 : Every finally stable $\mathbb{C}_2$-net contains a stable $\mathbb{C}_2$-subnet.

Theorem 3.10 : The given inferences can also be proved:

(i) $\mathbb{C}_0$-point confluence $\mathbb{C}_2$-net is $\mathbb{C}_0$(L)-confluence.

(ii) $\mathbb{C}_0$(L)-confluence $\mathbb{C}_2$-net is $\mathbb{C}_0$(L)-confluence.

(iii) $\mathbb{C}_0$(P)-confluence $\mathbb{C}_2$-net is $\mathbb{C}_0$(F)-confluence.

Proof : The $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is $\mathbb{C}_2$-point confluenced to $\eta = a_1 + i_1a_2 + i_2a_3 + ja_4$, then for every $\epsilon > 0$, we have

$$M_\epsilon = (a_1 + i_1a_2 + i_2a_3 + j(a_4 - \epsilon),\, a_1 + i_1a_2 + i_2a_3 + j(a_4 + \epsilon)_{\mathbb{C}_0} \in G_4.$$

Then for every $\gamma \in D$, such that $\eta_\alpha \in M_\epsilon, \forall \alpha \geq \gamma$. Further, for every $N \in G_3$, $M_\epsilon \subset G_3$. Therefore, the $\mathbb{C}_2$-net is also, $\mathbb{C}_0$(L)-confined to R-line $[x_1 = a_1, x_2 = a_2, x_3 = a_3]_{\mathbb{C}_0}$. Similarly, it can be shown that the $\mathbb{C}_2$-net which is $\mathbb{C}_0$(L)-confluence to the R-line $[x_1 = a_1, x_2 = a_2, x_3 = a_3]_{\mathbb{C}_0}$ is $\mathbb{C}_0$(P)-confluence to the R-plane $[x_1 = a, x_2 = b]_{\mathbb{C}_0}$ and the $\mathbb{C}_2$-net which is $\mathbb{C}_0$(P)-confluence to the R-plane $[x_1 = a_1, x_2 = a_2]_{\mathbb{C}_0}$ is $\mathbb{C}_0$(F)-confluence to the R-frame $[x_1 = a_1]_{\mathbb{C}_0}$. □

The above implications are non-reversible until and unless the $\mathbb{C}_2$-net is stable or finally stable.

Example 3.11 : On the directed set $(\mathbb{Q}^+, \geq)$, construct a $\mathbb{C}_2$-net $\{\eta_\beta\}$ as follows:

$$\{\eta_\beta\} = (k - y_\alpha) + (1/\beta)i_1 + (\beta + 1)i_2 + \beta\, j, \ \ \forall\, \alpha \in \mathbb{Q}^+$$

where the net $\{y_\alpha\}$ on $\mathbb{C}_0$ is finally stable on 0. Therefore, $\{y_{1\alpha}\}$ is finally stable at 'k' and $\{y_{2\alpha}\}$ converges on 0. So, the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is $\mathbb{C}_0$(P)-confluence to $[x_1 = k, x_2 = \ell]_{\mathbb{C}_0}$, for some $\ell \in \mathbb{C}_0$. Since, the net $\{y_{1\alpha}\}$ is finally stable at 'k'. Then, the $\mathbb{C}_2$-net $\{\eta_\alpha\}$ is finally in $N \in G_1$ for which $[x_1 = k]_{\mathbb{C}_0} \subset N$. Hence, $\{\eta_\alpha\}$ is $\mathbb{C}_0$(F)-confluence $[x_1 = k]_{\mathbb{C}_0}$.

Example 3.12 : For the directed set $(\mathbb{Q}^+, \geq)$, consider a $\mathbb{C}_2$-net $\{\eta_\beta\}$ as given below:

$$\{\eta_\beta\} = 2 - \frac{1}{\beta + 1} + i_1\beta + i_2\beta^2 + j, \ \ \forall\, \beta \in \mathbb{Q}^+$$

This $\mathbb{C}_2$-net is $\mathbb{C}_0$(F)-confluence to the R-frame $[x_1 = 2]_{\mathbb{C}_0}$. Then the net $\{y_{1\beta}\}$ converges to 2, but the net $\{y_{2\beta}\}$ is not convergent. Therefore, the $\mathbb{C}_2$-net $\{\eta_\beta\}$ is not $\mathbb{C}_0$(P)-confined to any R-plane contained in the R-frame $[x_1 = 2]_{\mathbb{C}_0}$.

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