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STRONG RATE OF CONVERGENCE OF EULER-MARUYAMA SCHEME FOR SDEs WITH SUPERCRITICAL COEFFICIENTS

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Abstract

Recently much attention has been paid to the strong rate of convergence of Euler-Maruyama scheme for SDEs with irregular coefficients, driven by α-stable-type processes. However, most of the studies carried out so far deal with the case where $\alpha \in (1,2)$. There are few works, if any, on the case where $\alpha \in (0,1]$. The aim of this paper is to contribute in narrowing this gap. Consider the stochastic differential equation (SDE)

$$\begin{aligned} dX_t &= x + b(t,X_t)dt + \int_0^\infty \int_{|z|\leq 1} 1_{[0,\ \sigma(s,X_{s^-},z)]}(r) z \tilde{\mathcal{N}}(dz,dr,dt) \\ &\quad + \int_0^\infty \int_{|z|>1} 1_{[0,\ \sigma(s,X_{s^-},z)]}(r) z \mathcal{N}(dz,dr,dt), \qquad X_0 = x \in \mathbb{R}^d. \end{aligned} \tag{0.1}$$

This SDE is a more general equation than the standard SDEs usually used in the literature on the strong rate of convergence of the Euler-Maruyama scheme. Furthermore, the driving process here is assumed to be an α-stable-type process, with $\alpha \in (0,1]$. We determine the strong rate of convergence of the Euler-Maruyama scheme for the SDE (0.1) under weak conditions on the coefficients. For example, the drift coefficient b is simply required to be a function in Besov spaces, which is

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a weaker condition than, but covers, the Hölder continuity requirement. This work thus extends many of the recent studies on the strong rate of convergence of Euler-Maruyama scheme for SDEs with irregular coefficients and driven by α-stable-type processes. Our method is based on the properties of the solution of the backward Kolmogorov equation associated with this SDE.

1. Introduction

Consider the stochastic differential equation

$$\begin{aligned} dX_t &= x + b(t, X_t)dt + \int_0^\infty \int_{|z|\leq 1} 1_{[0,\ \sigma(s,X_{s^-},z)]}(r) z \tilde{\mathcal{N}}(dz, dr, dt) \\ &\quad + \int_0^\infty \int_{|z|>1} 1_{[0,\ \sigma(s,X_{s^-},z)]}(r) z \mathcal{N}(dz, dr, dt), \quad X_0 = x \in \mathbb{R}^d. \end{aligned} \tag{1.1}$$

where $b : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a Borel function and $\sigma : \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d \longrightarrow \mathbb{R}$ is the jump intensity kernel. $\mathcal{N}(dz, dr, dt)$ is a Poisson measure on $\mathbb{R}^d \times [0,\infty) \times [0,\infty)$ and $\tilde{\mathcal{N}}(dz, dr, dt) := \mathcal{N}(dz, dr, dt) - \nu(dz)drdt$, the corresponding Poisson random measure where ν is a Lévy measure. Note that, here, the jump intensity kernel $\sigma(t, x, z)$ is state-dependent and the the driving process is a markov process, which is not necessarily a Lévy motion. The SDE (1.1) is thus a more general SDE than the standard SDEs often considered in the literature. More details on this SDE can be found in [15].

The solution of (1.1), when it exists, cannot always be found through analytical means. Often one has to resort to numerical methods. Our reference for numerical methods for SDEs include the books [6, 13]. The simplest and most used numerical scheme for approximating the solutions of SDEs remains the explicit Euler-Maruyama scheme (see Section 2.4). The aim of this paper is to determine the strong rate of convergence of the Euler-Maruyama scheme for the SDE (1.1) under mild conditions on the drift function b. Our method is based on regularity properties of the solution of the parabolic partial differential equation

$$\partial_t u + \mathcal{L}_\nu^\sigma u - \lambda u + b.\nabla u = -f, \quad u(0,x) = 0. \tag{1.2}$$

where $\lambda > 0$ is a parameter and $b, f : [0,1] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ are Borel functions. Here the operator $\mathcal{L}_\nu^\sigma$ is defined by

$$\mathcal{L}_\nu^\sigma f(x) = \int_{\mathbb{R}^d} \big(f(x+z) - f(x) - 1_{\{\alpha \geq 1\}} 1_{\{|z|\leq 1\}} z.\nabla f(x) \big)\, \sigma(t,x,z)\nu(dz) \tag{1.3}$$

There are many interesting results on the strong rate of convergence of the Euler-Maruyama approximation for SDEs which allow non-globally Lipschitz coefficients. We

refer to [4, 12, 8] and the references therein for a review of the work done on the strong rate of convergence of the Euler-Maruyama approximations of SDEs driven by a Wiener motion and to [14, 5, 3, 8, 10, 7] and the references therein for more information on the work done on the strong rate of convergence of the Euler-Maruyama approximation to the solution of SDEs with jumps.

Let us remark that the field of partial differential equations (PDEs) provides useful tools for investigating the rate of convergence of a numerical scheme. For example, Mikulevičius and Platen in [9] used the backward Kolmogorov equation to study the weak rate of convergence of the Euler Maruyama scheme when the coefficients are β-Hölder continuous. The results of [9] were generalised in [11] to the case of non-degenerate SDEs driven by Lévy processes. However, Pamen and Taguchi [8] seem to be the first to use the backward Kolmogorov equations to estimate the strong rate of convergence of SDEs with irregular coefficients. Since then the idea has been used by many authors to investigate the strong convergence and the strong rate of convergence of the Euler-Maruyama scheme. In the continuous case, Bao et al.(2016) [1] has used the idea to extend the result of [8] to the case of SDEs with Hölder-Dini continuous coefficients and possibly unbounded drift. Dareiotis and Gerencsér (2018) also used this approach. In the case of SDEs with jumps, Mikulevičius and Xu [10] extended the result of [8] to the case of SDEs with non constant diffusion using this same technique. Very recently, using the same idea, Kühn and Schilling [7] extended the result of [8] to the case where the driving process L_t belongs to a wide class of Lévy processes.

We note that most of the research on the strong rate of convergence of the Euler-Maruyama approximation to the solution of SDEs with jumps has focused on sub-critical SDEs. There are much fewer studies on the supercritical SDEs in the literature. Motived by the recent work of [15], we are going to use the PDE approach introduced in [8] to estimate the strong rate of convergence the Euler-Maruyama approximation to the solution to the SDE (1.1) under some mild conditions on the coefficients of (1.1). More specifically, we assume that the drift coefficient b is a function in the Besov space $B^{\beta}_{p',q'}(\mathbb{R}^d)$. For information on functions in the non-homogeneous Besov spaces, one may consult [15]. The present work thus extends the works of [8, 7, 10].

Notations and Assumptions:

We adopt the notations and assumptions of [15, 8]. We use the standard notation ∇f

to denote the vector of partial derivatives with respect to the space variable and $D_t f$ to indicate the time derivative of f. For $a, b, R > 0 \in \mathbb{R}$, we use $a \wedge b$ for $\min\{a, b\}$, $a \vee b$ for $\max\{a, b\}$, a^+ for $\max\{a, 0\}$ and B_R for the ball of centre 0 and radius R, i.e., $B_R := \{x \in \mathbb{R}^d : |x| \leq R\}$. If $\mathbf{F}$ is a class of functions then $C([a, b], \mathbf{F})$ denotes the space of all functions $f : [a, b] \times \mathbb{R}^d \longrightarrow \mathbb{R}^k$ such that $f(t, .) \in \mathbf{F}$ for all $t \in [a, b]$. We mention some function spaces:

- $C_b(\mathbb{R}^d; \mathbb{R}^k)$ denotes the space of bounded continuous functions $f : \mathbb{R}^d \longrightarrow \mathbb{R}^k$. For a measurable function f, the supremum norm is defined $\|f\|_\infty := \sup_{x \in \mathbb{R}^d} |f(x)|$.

- If $\beta \in (0, 1]$ and $f : [0, T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a function, we define the Hölder semi-norm by
$$[f]_\beta := \sup_{x \neq y} \frac{|f(x) - f(y)|}{|x - y|^\beta};$$
and
$$\|f\|_{L_T^\infty C_b^\beta} := \sup_{(t,x) \in [0,T] \times \mathbb{R}^d} |f(t, x)| + \sup_{t \in [0,T]} [f(t)]_\beta$$

- For $p' \in [1, \infty]$, we write $L_{p'}^\infty(T) := L^\infty([0, T]; L^{p'}(\mathbb{R}^d))$ with norm
$$\|f\|_{L_{p'}^\infty(T)} := \sup_{t \in [0,T]} \|f(t, .)\|_{p'}$$

- Given $\beta \in \mathbb{R}$ and $p', q' \in [1, \infty]$, we denote the non-homogeneous space of Besov space by $B_{p',q'}^\beta(\mathbb{R}^d))$. More information on Besov spaces can be found in [15] and the references therein. If $f \in L^\infty([0, T]; B_{p',q'}^\beta(\mathbb{R}^d))$, then the norm of f is given by
$$\|f\|_{L_T^\infty B_{p',q'}^\beta} := \sup_{t \in [0,T]} \|f(t, .)\|_{B_{p',q'}^\beta}.$$

- For $0 < \alpha < 2$, we denote the set of all non-degenerate non local symmetric stable Lévy measures ν^α by $\mathbb{L}^\alpha$.

We adopt the following assumptions of [15].

Assumption 1.1 : Assumption on the Lévy measure: we assume that

- (H_0^ν): the Lévy measure ν is of α-stable type with $\alpha \in (0, 1]$,

- (H_1^ν): $\nu(B_1^c) < \infty$ and there exist $\nu_1,\ \nu_2 \in \mathbb{L}^\alpha$ such that

$$\nu_1(A) \leq \nu(A) \leq \nu_2(A), \qquad \forall A \subseteq B_1,$$

- H_2^ν : there exist $\gamma \in [0,1]$ and $\gamma_\infty > 0$ such that

$$\int_{|z|\leq 1} |z|^{1+\gamma} \nu(dz) < \infty \qquad and \qquad \int_{|z|>1} |z|^{\gamma_\infty} \nu(dz) < \infty.$$

Assumption 1.2 : Regularity Assumptions on the drift b: We assume that the drift coefficient b is a function i $L^\infty(\mathbb{R}^+; B^\beta_{p',\infty})$ where

$$\beta > 1 - \frac{\alpha}{2} \qquad and \qquad \frac{2d}{\alpha} < p' \leq \infty.$$

Assumption 1.3 : Regularity Assumptions on σ: assume that

- (H_1^σ) : there exist constants $\kappa_0, \kappa_1 > 0$, $\kappa_2 \geq 1$ such that

$$\kappa_0 \leq \sigma(t,x,z) \leq \kappa_1, \qquad \forall\ (t,x,z) \in \mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{R}^d$$

  and for all $(t,z) \in \mathbb{R}^+ \times \mathbb{R}^d$ and all $\theta \in (0,1]$ we have

$$|\sigma(t,x,z) - \sigma(t,y,z)| \leq \kappa_2 |x-y|^\theta,\ with\ |x-y| \leq 1.$$

- (H_2^σ) : there exists a function $\varrho \in B^0_{q',\infty}(\mathbb{R}^d)$ with $q' > \frac{d}{\alpha}$ such that for every $t > 0$ and almost all $x, y \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |\sigma(t,x,z) - \sigma(t,y,z)|\, (|z| \wedge 1) \nu(dz) \leq |x-y|(\varrho(x) + \varrho(y)).$$

Under these assumptions, the existence and uniqueness of the strong solution of the SDE (1.1) are guaranteed by [15, Theorem 2.6]. The objective of this work is to determine the strong rate of convergence of the Euler-Maruyama approximation to the solution of the SDE (1.1) under the above assumptions. As stated above, the condition $b \in L^\infty(\mathbb{R}^+; B^\beta_{p',\infty})$ is weaker than the condition $b \in C_b^\beta(\mathbb{R}^d)$ imposed on the drift in, e.g., [8, 10, 7].

The remaining part of this work is organized as follows. In section 2 we gather preliminary results necessary for the proof of the results of this work. In section 3, we state the main results of this paper. Finally, in section 4, we prove the main theorems of this

paper.

2. Preliminaries

2.1 Moments estimates for the Process : Let $(\Omega, \mathcal{F}, P)$ be a probability space. A process $Z_t : \Omega \longrightarrow \mathbb{R}^d$ is said to be a stable type process if Z_t is a pure jump process the jump intensity kernel of which is comparable to that of one or more stable processes (see [15]. Throughout this section we assume that ν is a Lévy measure of α-stable type. We set

$$Z_t^1 := \int_0^t \int_0^\infty \int_{|z|\leq 1} 1_{[0,\ \sigma(s,x,z)]}(r) z \tilde{\mathcal{N}}(dz, dr, dt)$$

and

$$Z_t^2 := \int_0^t \int_0^\infty \int_{|z|\leq 1} 1_{[0,\ \sigma(s,x,z)]}(r) z \tilde{\mathcal{N}}(dz, dr, dt).$$

The moments estimates of Z_t^1 and Z_t^2 play an important role in the proof of the main results of this work. Let $Z_t = Z_t^1 + Z_t^2$. The following Lemma is essential for this study.

Lemma 2.1 : Assume that the assumptions (1.1) and (1.3) hold. Then for any $p > 0$, there exists a constant C depending on $p, d, \kappa_1, \gamma, \gamma_\infty, \kappa_1, \nu_2(B_1), \nu(B_1^c)$ such that for any $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{S\leq t\leq T} |Z_t|^p\right] \leq \begin{cases} Ct^{\frac{p}{2}} & if \qquad p > 1 \\ Ct^{\frac{p}{\alpha}} & if \qquad p < \alpha \in (0, 1] \end{cases}$$

Proof : We prove Lemma 2.1 for the constant coefficients case $\sigma = \sigma_0(s, z)$. The prove for the variable coefficients case is similar. Using Doob's maximal inequality and assumption (1.1), it follows that

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T} |Z_t^1|^p\right] &\leq C_p \mathbb{E}\left[\left(\int_0^t \int_0^\infty \int_{|z|\leq 1} 1_{[0,\ \sigma(s,z)]}(r)|z|^2 dr\nu(dz)ds\right)^{\frac{p}{2}}\right] \\ &\leq C_p \mathbb{E}\left[\left(\int_0^t \int_{|z|\leq 1} \sigma(s,z)|z|^\gamma \nu(dz)ds\right)^{\frac{p}{2}}\right] \\ &\leq C_p \left(\kappa_1 \int_{|z|\leq 1} |z|^\gamma \nu(dz)\right)^{\frac{p}{2}} \left(\int_0^t ds\right)^{\frac{p}{2}} \leq C_{p,\kappa_1,C_{\nu,\gamma}} t^{\frac{p}{2}} \end{aligned} \tag{2.1}$$

with a similar inequality for the process Z_t^2. Thus for any $p \geq 1$ there exists $C =$

$C(p,\kappa_0,\kappa_1)$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}|Z_t|^p\right] \;\le\; C_{p,\kappa_1,\kappa_0,\gamma,\gamma_\infty}t^{\frac{p}{2}}$$

In the case where $p\in(0,1)$ we use [[7], Theorem 3.1(i)]:

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}|Z_t^1|^p\right] &:= \mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\int_0^\infty\int_{|z|\le 1}1_{[0,\ \sigma(s,z)]}(r)z\tilde{\mathcal{N}}(dz,dr,dt)\right|^p\right]\\ &\le C_p\mathbb{E}\left[\left(\int_0^t\int_0^\infty\int_{|z|\le 1}1_{[0,\ \sigma(s,z)]}(r)|z|^\alpha dr\nu(dz)ds\right)^{\frac{p}{\alpha}}\right]\\ &\le C_p\mathbb{E}\left[\left(\int_0^t\int_{|z|\le 1}\sigma(s,z)|z|^\alpha\nu(dz)ds\right)^{\frac{p}{\alpha}}\right]\le C_{p,\kappa_1,\alpha,\nu_2(B_1)}t^{\frac{p}{\alpha}}\end{aligned}$$

with a similar inequality for the process Z_t^2. Thus for all $p\in(0,1)$, we have

$$\mathbb{E}\left[\sup_{0\le t\le T}|Z_t|^p\right] \;\le\; C_{p,\kappa_1,\kappa_0,\nu_1(B_1^c),\nu_2(B_1)}t^{\frac{p}{\alpha}}$$

The proof is complete. □

2.2 The PDE associated with SDE (1.1) : The method we are going to use for determining the strong rate of convergence of the Euler Maruyama approximations relies heavily on the regularity properties of the solution of the backward Kolmogorov equation associated with the SDE. Following [15], we consider the linear parabolic partial differential equation

$$\partial_t u+\mathcal{L}_\nu^\sigma u-\lambda u+b.\nabla u=-f,\qquad u(0,x)=0. \tag{2.2}$$

where $\mathcal{L}_\nu^\sigma$ is the infinitesimal generator defined by (1.3), $\lambda>0$ is a parameter and $b,f:[0,1]\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ are Borel functions. In this paper we refer to the partial differential equation (2.2) as the backward Kolmogorov equation associated with SDE (1.1). The existence, uniqueness and properties of the strong solution to the integro differential equation (2.2) are studied in [[15], Theorems 4.3, 4.6, 4.8 and 4.9]. The following Lemma is a direct consequence of [[15], Theorem 4.6].

Lemma 2.2 : Assume that $0<\alpha\le 1$ and let hypothesis (H_1^σ) hold. Suppose that $b\in L^\infty([0,T];B^\beta_{p',\infty}(\mathbb{R}^d))$ with $\beta>1-\alpha$ and $\frac{d}{\alpha+\beta-1}\vee 2<p'\le\infty$. Let $\lambda\ge 0$ and

$f \in L^\infty([0,T];B^\gamma_{q',\infty}(\mathbb{R}^d))$ with $\gamma \in [0,\ \beta]$, $2 \leq q' \leq p'$ and $q' < \infty$. Then for all $\varepsilon \in (0,1)$, the unique solution $u \in L^\infty([0,T];B^{\alpha+\gamma}_{q',\infty}(\mathbb{R}^d))$ to the PDE

$$\partial_t u + (\mathcal{L}^\sigma_\nu - \lambda)u + b.\nabla u = -f, \qquad u(0,x) = 0.$$

satisfies the inequality

$$\|\mathrm{u}\|_\infty + \|\nabla u\|_\infty \leq \varepsilon \leq \frac{1}{2}. \tag{2.3}$$

Proof : By [[15], Theorem 4.3 and 4.6] for all $\theta \in [0, \alpha+\gamma)$, the unique solution, $u \in L^\infty([0,T];B^{\alpha+\gamma}_{q',\infty}(\mathbb{R}^d))$ to the PDE (2.2) satisfies

$$\|u(t,.)\|_{L^\infty_T B^\theta_{q',\infty}} \leq C_\lambda \|f(t,.)\|_{L^\infty_T B^\gamma_{q',\infty}}. \tag{2.4}$$

for some constant C_λ such that $C_\lambda \longrightarrow 0$ as $\lambda \longrightarrow \infty$. Since $\lambda > 0$ in [[15], Theorem 4.3] is arbitrary, it can be selected so that $C_\lambda \leq \frac{\varepsilon}{\|f(t,.)\|_{L^\infty_T B^\gamma_{q',\infty}}} \leq \frac{1}{2}$. Since $L^\infty_{p'}(\mathbb{R}^d) \subset B^0_{q',\infty}$, the result then follows from the embedding Theorem given in [15], i.e.,

$$B^{\beta+\frac{d}{p'}}_{p',q'}(\mathbb{R}^d) \hookrightarrow C^\beta_b(\mathbb{R}^d) \tag{2.5}$$

The proof is complete. □

The following corollary give the properties of the solutions of the PDE (2.2) in the case $q' = \infty$. In this case $B^\beta_{\infty,\infty}(\mathbb{R}^d)$ is a Hölder space.

Corollary 2.3 : Assume that $0 < \alpha \leq 1$. Assume that b is bounded and β-Hölder continuous in x with $\beta \in (0,1)$ and $\alpha + \beta > 1$. Let $\lambda > 0$, and $f \in L^\infty([0,T);C^\gamma_b(\mathbb{R}^d))$ with $\gamma \in [0,\ \theta \wedge \beta]$. Then, for all $\varepsilon \in (0,1)$, the unique solution $u \in L^\infty([0,T];B^{\alpha+\gamma}_{\infty,\infty}(\mathbb{R}^d))$ to the PDE

$$\partial_t u + (\mathcal{L}^\sigma_\nu - \lambda)u + b.\nabla u = -f, \qquad u(0,x) = 0.$$

satisfies the inequality

$$\|\mathrm{u}\|_\infty + \|\nabla u\|_\infty \leq \varepsilon \leq \frac{1}{2}. \tag{2.6}$$

Proof : It is enough to let $q = \infty$ in Lemma 2.2 and use the fact that $B^\beta_{\infty,\infty}(\mathbb{R}^d) = C^\beta_b(\mathbb{R}^d)$ (see [15]). □

2.3 The transformed equation : Let u be the solution of the backward Kolmogorov equation (2.2) with f replaced by b, i.e.,

$$\partial_t u + (\mathcal{L}^\sigma_\nu - \lambda)u + b.\nabla u = -b, \qquad u(0,x) = 0.$$

For any $t \in [0,T]$, consider the mapping $\mathcal{T} : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ defined by $\mathcal{T}_t(x) = x + u(t,x)$. Then $\mathcal{T}$ is C^1-diffeomorphism $\mathcal{T}^{-1}$, with

$$\frac{1}{2} \leq \|\nabla \mathcal{T}\|_\infty, \quad \|\nabla \mathcal{T}\|_\infty^{-1} \leq 2. \tag{2.7}$$

(see [15]). We now introduce a new SDE which will play an important role in the study of the strong rate of convergence of SDE (1.1). Let X_t denote the solution of the SDE (1.1). Then, by [[15], Lemma 5.5], the process $Y_t := \mathcal{T}_t(X_t) = X_t + u(t, X_t)$ is a strong solution to the SDE

$$\begin{aligned} Y_t &= \mathcal{T}_t(x) + \int_0^t \tilde{b}(s, Y_s)ds + \int_0^t \int_0^\infty \int_{|z|\leq 1} 1_{[0,\tilde{\sigma}(s,z)]}(r) z \tilde{\mathcal{N}}(dz, dr, ds) \\ &\quad + \int_0^t \int_0^\infty \int_{|z|>1} 1_{[0,\tilde{\sigma}(s,z)]}(r) z \mathcal{N}(dz, dr, dt) \end{aligned} \tag{2.8}$$

where

$$\tilde{b}(t,x) := \lambda u(t, \mathcal{T}_t^{-1}(x)) - \int_{|z|>1} \left(u(t, \mathcal{T}_t^{-1}(x) + z) - u(t, \mathcal{T}_t^{-1}(x)) \right) \tilde{\sigma}(t,z) \nu(dz) \tag{2.9}$$

and

$$g_t(x,z) := \mathcal{T}_t(\mathcal{T}_t^{-1}(x) + z) - x, \qquad \tilde{\sigma}(t,z) := \sigma(t,z). \tag{2.10}$$

In this paper we refer to the SDE (2.8) as the auxiliary SDE associated with SDE (1.1). The advantage of the auxiliary SDE (2.8) over the original SDE (1.1) is that it has more regularity. The coefficients $\tilde{b}$ for example satisfies the following conditions.

Lemma 2.4 : Let the hypotheses Lemma 2.2 hold. Then the coefficients $\tilde{b}$ in the transformed equation (2.8) satisfy the inequality

$$\left| \tilde{b}(t,x) - \tilde{b}(t,y) \right| \leq \varepsilon \left(\lambda + \kappa_2 \nu(B_1^c) \right) |x - y|$$

in the constant coefficients case where $\sigma = \sigma_0(t,z)$, and

$$\left| \tilde{b}(t,x) - \tilde{b}(t,y) \right| \leq 2\varepsilon \left(\lambda + \varrho(\mathcal{T}^{-1}(x)) + \varrho(\mathcal{T}^{-1}(y)) \right) |y - x|$$

in the variable coefficients case, $\sigma = \sigma(t,x,z)$

Proof : We use the MVT, hypothesis H_2^σ and Lipschitz properties of the mappings $\mathcal{T}_t$ and $\mathcal{T}_t^{-1}$. In the constant coefficients case we have

$$\begin{aligned}\tilde{b}(t,x)-\tilde{b}(t,y) &:= \lambda u(t,\mathcal{T}_t^{-1}(x))-\lambda u(t,\mathcal{T}_t^{-1}(y))\\ &\quad-\int_{|z|>1}\left(u(t,\mathcal{T}_t^{-1}(x)+z)-u(t,\mathcal{T}_t^{-1}(x))\right)\tilde{\sigma}(t,z)\nu(dz)\\ &\quad+\int_{|z|>1}\left(u(t,\mathcal{T}_t^{-1}(y)+z)-u(t,\mathcal{T}_t^{-1}(y))\right)\tilde{\sigma}(t,z)\nu(dz).\end{aligned}$$

Thus

$$\begin{aligned}\left|\tilde{b}(t,x)-\tilde{b}(t,y)\right| &\le \lambda\left|u(t,\mathcal{T}_t^{-1}(x))-u(t,\mathcal{T}_t^{-1}(y))\right|\\ &\quad+\varepsilon\left|\mathcal{T}_t^{-1}(x)-\mathcal{T}_t^{-1}(y)\right|\int_{|z|>1}\tilde{\sigma}(t,z)\nu(dz)\le 2\varepsilon\left(\lambda+\kappa_2\nu(B_1^c)\right)|x-y|.\end{aligned}$$

where we have used (2.7). In the variable coefficient case, we have

$$\begin{aligned}\left|\tilde{b}(t,x)-\tilde{b}(t,y)\right| &\le \lambda\varepsilon\left|\mathcal{T}^{-1}(x)-\mathcal{T}^{-1}(y)\right|\\ &\quad+\varepsilon\int_{|z|>1}\left|\sigma(t,\mathcal{T}^{-1}(y),z)-\sigma(t,\mathcal{T}^{-1}(x),z)\right|(|z|\wedge 1)\nu(dz)\\ &\le \lambda\varepsilon\left|\mathcal{T}^{-1}(x)-\mathcal{T}^{-1}(y)\right|+\varepsilon\left|\mathcal{T}^{-1}(y)-\mathcal{T}^{-1}(x)\right|\left(\varrho(\mathcal{T}^{-1}(x))+\varrho(\mathcal{T}^{-1}(y))\right)\\ &\le 2\varepsilon\left(\lambda+\varrho(\mathcal{T}^{-1}(x))+\varrho(\mathcal{T}^{-1}(y))\right)|y-x|.\end{aligned}$$

where we have used the hypothesis (H_2^σ). The proof is complete. □

2.4 Euler-Maruyama Approximation : Suppose there exists a probability space $(\Omega,\mathcal{F},P)$ on which one can define a stable-type process Z_t and a process X_t such that the SDE (1.1) is satisfied. Then the continuous Euler-Maruyama scheme for (1.1) is given by

$$\begin{aligned}X_t^{(n)} &= x+\int_0^t b(s,X_{\eta_n(s)}^{(n)})ds+\int_0^t\int_0^\infty\int_{|z|\le 1}1_{[0,\ \sigma(s,X_{\eta(s)}^{(n)},z)]}(r)z\tilde{\mathcal{N}}(dz,dr,ds)\\ &\quad+\int_0^t\int_0^\infty\int_{|z|>1}1_{[0,\ \sigma(s,X_{\eta(s)}^{(n)},z)]}(r)z\mathcal{N}(dz,dr,ds)\end{aligned}\tag{2.11}$$

where $\eta(s)=\frac{kT}{n}, s\in\left[\frac{kT}{n},\frac{(k+1)T}{n}\right]$. The following Lemma plays an important role in the proof of the main result.

Lemma 2.5 : Let the hypotheses of assumptions (1.2), (1.3) hold. Suppose, in addition, that b is bounded. Then for any $p\ge 0$ there exists a constant C depending on

$p, d, T, \|b\|_{L_{p'}^{\infty}(T)}$ such that

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{(n)} - X_{\eta_n(t)}^{(n)}\right|^p\right] \le \begin{cases} \frac{C}{n^{\frac{p}{2}}} & if \qquad p>1 \\ \frac{C}{n^{\frac{p}{\alpha}}} & if \qquad p\le\alpha\in(0,1]. \end{cases}$$

Proof : Using [[15], Theorem 2.6] and applying Jensen's and Hölder inequalities and taking supremum and expectation, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t^{(n)} - X_{\eta_n(t)}\right|^p\right] \\
&\le 3^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}(t-\eta_n(t))^{p-1}\int_{\eta_n(t)}^{t}\sup_{0\le u\le s}\left|b(u, X_{\eta_n(u)}^{(n)})\right|^p ds\right] \\
&+ 3^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_{\eta_n(t)}^{t}\int_0^{\infty}\int_{|z|\le 1} 1_{[0,\sigma(s,X_s^{(n)},z)]}(r) z\tilde{\mathcal{N}}(dz,dr,dt)\right|^p\right] \\
&+ 3^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_{\eta_n(t)}^{t}\int_0^{\infty}\int_{|z|> 1} 1_{[0,\sigma(s,X_s^{(n)},z)]}(r) z\mathcal{N}(dz,dr,dt)\right|^p\right] \\
&\le 3^{p-1}T^p\frac{\|b\|_{L_p^{\infty}}^p}{n^p} + 3^{p-1}\mathbb{E}\left[\sup_{0\le t\le T}\left|Z_{t-\eta_n(s)}\right|^p\right]
\end{aligned}$$

where $Z_t = Z_t^1 + Z_t^2$ as defined in subsection (2.1). The result follows from Lemma (2.1). □

The following Lemma is essential for the prove of the results of this work.

Lemma 2.6 : Assume that the hypotheses of assumptions (1.1) - (1.3) hold. Then for any $p>0$, there exists $C>0$ such that for any $t\in[0,T]$, there exists a constant C such that for all $n\in\mathbb{N}$

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|X_t - X_t^{(n)}\right|^p\right] \quad\le\quad C\mathbb{E}\left[\sup_{0\le t\le T}\left|Y_t - \mathcal{T}(X_t^{(n)})\right|^p\right].$$

Proof : Since the mapping $\mathcal{T}^{-1}$ is Lipschitz continuous, it holds that

$$\left|X_t - X_t^{(n)}\right| := \left|\mathcal{T}^{-1}(Y_t) - \mathcal{T}^{-1}(\mathcal{T}(X_t^{(n)}))\right| \le C\left|Y_t - \mathcal{T}(X_t^{(n)})\right|.$$

Now the result follows by taking the supremum and then expectation on both sides of the above inequality. □

3. Main results

The following theorems constitute the result of this paper. Theorem 3.1 determines the strong rate of convergence of the Euler-Maruyama approximation $X_t^{(n)}$ to the solution of (1.1) in the case where b is Hölder continuous.

Theorem 3.1 : Assume that $0 < \alpha \leq 1$ and the hypotheses H_1^σ and H_2^σ hold. Assume that b is bounded, η-Hölder continuous in the time variable and β-Hölder continuous in x with $\beta \in (0, 1]$ and $\alpha + \beta > 1$. Then there exists a constant C depending on $p, d, T, K, \varepsilon, \nu(B_1), \|b\|_{L^\infty_{p'}(T)}$ and $\|\varrho\|_{L^\infty_T B^0_{q',\infty}}$ such that

$$\mathbb{E}\left[\sup_{0\leq\leq T}\left|X_t - X_t^{(n)}\right|^p\right] \leq \begin{cases} \frac{C}{n^{p\eta\wedge\frac{p\beta}{2}}} & \text{if} \quad p\beta > 1. \\ \frac{C}{n^{p\eta\wedge\frac{p\beta}{\alpha}}} & \text{if} \quad p\beta < \alpha. \end{cases}$$

The following corollary is a direct consequence of Theorem 3.1.

Corollary 3.2 : Under the assumptions of Theorem 3.1, there exists a constant C depending on $p, d, T, K, \varepsilon, \lambda, \nu(B_1), \|b\|_{L^\infty_{p'}(T)}$ and $\|\varrho\|_{L^\infty_T B^0_{q',\infty}}$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t - X_t^{(n)}\right|^p\right] \leq \begin{cases} \frac{C}{n^{\frac{p\beta}{2}}} & \text{if} \quad p\beta > 1 \ \wedge \ \frac{\beta}{2} \leq \eta < 1. \\ \frac{C}{n^{\frac{p\beta}{\alpha}}} & \text{if} \quad p\beta < \alpha \ \wedge \ \frac{\beta}{\alpha} \leq \eta < 1. \end{cases}$$

The following Theorem is our second main result. It deals with the case where the drift coefficients is not necessarily Hölder continuous.

Theorem 3.3 : Let $0 < \alpha \leq 1$. Assume that the hypotheses in Assumptions (1.1) - (1.3) all hold. Assume further that the spacial first partial derivatives of the drift function b exist and satisfy the assumption (1.2). Suppose in addition that b is bounded and η-Hölder continuous in the time variable. Then for any $p > 0$, there exists a positive constant C depending on $\lambda, d, p, T, |b|_{L^\infty_{p'}(T)}, |\nabla b|_{p'}, |\kappa|_2, \nu(B_1), \nu(B_1^c), \varepsilon,$ and $\varrho_{L^\infty_T B^0_{q',\infty}}$ such that for any $t \in [0, T]$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X_t - X_t^{(n)}\right|^p\right] \leq \begin{cases} \frac{C}{n^{\frac{p}{2}}} & \text{if} \quad p > 1, \ \alpha + \beta > 1 \ \wedge \ \eta \geq \frac{1}{2}, \\ \frac{C}{n^{\frac{p}{\alpha}}} & \text{if} \quad p \leq \alpha, \ \alpha + \beta > 1 \ \wedge \ \eta \leq \frac{1}{2}. \end{cases}$$

4. Proof of Main Results

We use the technique introduced by [8] which uses the regularity properties of the solution to the backward Kolmogorov equation associated with the SDE under consideration to derive the L^p-error of the scheme. More specifically, using Lemma 2.2, for

any $\varepsilon \in (0,1)$, and for any $i = 1, 2, ..., d$ the PDE

$$\frac{\partial u_i}{\partial t} + (\mathcal{L}_\nu^\sigma - \lambda)u_i + b.\nabla u_i = -b_i \ \text{ on } \qquad u_i(0,x) = 0 \tag{4.1}$$

has a unique strong solution, and moreover, for some λ large enough, we have

$$\|u_i\|_\infty + \|\nabla u_i\|_\infty \leq \varepsilon \leq \frac{1}{2}. \tag{4.2}$$

Applying the Itô formula (see [[15], Lemma 5.1]) to the solution u_i of the PDE (4.1) at X_t, we have

$$\begin{aligned} u_i(t,X_t) &= u_i(0,x) + \int_0^t (\partial_s + \mathcal{L}_s)u_i(s,X_s)ds \\ &\quad + \int_0^t \int_0^\infty \int_{\mathbb{R}^d} \left[u_i\left(s, X_{s^-} + 1_{[0,\ \sigma(s,X_{s^-},z)]}(r)z\right) - u_i(s,X_{s^-})\right] \tilde{\mathcal{N}}(dz \times dr \times ds) \\ &= u_i(0,x) + \lambda \int_0^t u_i(s,X_s)ds - \int_0^t b_i(s,X_s)ds \\ &\quad + \int_0^t \int_0^\infty \int_{\mathbb{R}^d} H(s,X_{s^-},z)\tilde{\mathcal{N}}(dz \times dr \times ds), \end{aligned}$$

where

$$\begin{aligned} H(s,x,r,z) &:= u_i\left(s, x + 1_{[0,\ \sigma(s,x,z)]}(r)z\right) - u_i(s,x) \\ &:= 1_{[0,\ \sigma(s,x,z)]}(r)\left(u_i\left(s,x+z\right) - u_i(s,x)\right). \end{aligned} \tag{4.3}$$

Thus

$$\begin{aligned} \int_0^t b_i(s,X_s)ds &= u_i(0,x) - u_i(t,X_t) + \lambda \int_0^t u_i(s,X_s)ds \\ &\quad + \int_0^t \int_0^\infty \int_{\mathbb{R}^d} H(s,X_{s^-},r,z)\tilde{\mathcal{N}}(dz \times dr \times ds). \end{aligned} \tag{4.4}$$

Similarly, applying the Itô formula to u_i at $X_t^{(n)}$ we have

$$\begin{aligned} \int_0^t b_i\left(s, X_s^{(n)}\right) ds &= \int_0^t \lambda u_i(s, X_s^{(n)})ds \\ &\quad + \int_0^t \left\langle \nabla u_i(s, X_{s^-}^{(n)}), \left(b(\eta_n(s), X_{\eta_n(s)}^{(n)}) - b(s, X_s^{(n)})\right)\right\rangle ds \\ &\quad + \int_0^t \int_0^\infty \int_{\mathbb{R}^d} H(s, X_{s^-}^{(n)}, r, z)\tilde{\mathcal{N}}(dz \times dr \times ds). \end{aligned} \tag{4.5}$$

Using the definitions (1.1), (2.11), (4.4) and (4.5), for each $i = 1, 2, \cdots, d$, we have

$$\begin{aligned}
&\left|X_t^{(i)} - X_t^{(n,i)}\right| \\
&\leq \left|u_i(t, X_t^{(n)}) - u_i(t, X_t)\right| + \left|\int_0^t \lambda(u_i(s, X_s) - u_i(s, X_s^{(n)}))ds\right| \\
&+ \left|\int_0^t \left(b_i(\eta_n(s), X^{(n)}_{\eta_n(s)}) - b_i(s, X_s^{(n)})\right) ds\right| \\
&+ \left|\int_0^t \left\langle \nabla u_i(s, X^{(n)}_{s^-}), \left(b(\eta_n(s), X^{(n)}_{\eta_n(s)}) - b(s, X_s^{(n)})\right)\right\rangle ds\right| \\
&+ \left|\int_0^t \int_0^\infty \int_{\mathbb{R}^d} K(s, X_{s^-}, X^{(n)}_{s^-}, r, z)\tilde{\mathcal{N}}(dz \times dr \times ds)\right| \\
&+ \left|\int_0^t \int_0^\infty \int_{|z|\leq 1} \left(1_{[0,\ \sigma(s, X_{s^-}, z)]}(r) - 1_{[0,\ \sigma(s, X^{(n)}_{\eta(s)^-}, z)]}(r)\right) z\tilde{\mathcal{N}}(dz, dr, ds)\right| \\
&+ \left|\int_0^t \int_0^\infty \int_{|z|>1} \left(1_{[0,\ \sigma(s, X_{s^-}, z)]}(r) - 1_{[0,\ \sigma(s, X^{(n)}_{\eta(s)^-}, z)]}(r)\right) z\tilde{\mathcal{N}}(dz, dr, ds)\right| \qquad (4.6)
\end{aligned}$$

where

$$K_i(s, X_{s^-}, X^{(n)}_{s^-}, r, z) := (H_i(s, X_{s^-}, r, z) - H_i(s, X^{(n)}_{s^-}, r, z)).$$

4.1 Proof of Theorem 3.1 : We first consider the case where the jump intensity kernel σ is space-independent. In this case the last two terms in (4.6) vanish. We need to control each of the terms on the right of the inequality (4.6). Since b is Hölder continuous in both variables, i.e. for all $s, t \in [0, T]$ and all $x, y \in \mathbb{R}^d$,

$$|b(t, x) - b(s, y)| \leq K(|s - t|^\eta + |x - y|^\beta),$$

we use the mean value Theorem (MVT) and Lemma 2.2 to reach the inequality:

$$\begin{aligned}
\left|X_t^{(i)} - X_t^{(n,i)}\right| \quad \leq \quad & \varepsilon \left|X_t^{(n)} - X_t\right| + \lambda\varepsilon \int_0^t \left|X_s - X_s^{(n)}\right| ds \\
& + (1 + \varepsilon)K \int_0^t \left|X^{(n)}_{\eta_n(s)} - X_s^{(n)}\right|^\beta ds + (1 + \varepsilon)K(t - T_{j-1})|s - \eta_n(s)|^\eta \\
& + \left|\int_0^t \int_0^\infty \int_{\mathbb{R}^d} K_i(s, X_{s^-}, X^{(n)}_{s^-}, r, z)\tilde{\mathcal{N}}(dz \times dr \times ds)\right|. \qquad (4.7)
\end{aligned}$$

Next, we form the L^p-norm of the error $\left|X_t - X_t^{(n)}\right|$. To this end, we use Jensen and

Hölder's inequalities:

$$\begin{aligned}
\left|X_t - X_t^{(n)}\right|^p &= \left(\sum_{i=1}^d \left|X_t^i - X_t^{(n,i)}\right|^2\right)^{\frac{p}{2}} = d^{\frac{p}{2}-1}5^{p-1}\sum_{i=1}^d \left|X_t^i - X_t^{(n,i)}\right|^p \\
&\leq d^{\frac{p}{2}-1}6^{p-1}\varepsilon^p \left|X_t^{(n)} - X_t\right|^p + (\lambda\varepsilon)^p T^{p-1}\int_{T_{j-1}}^t \left|X_s - X_s^{(n)}\right|^p ds \\
&\quad + d^{\frac{p}{2}-1}6^{p-1}(1+\varepsilon)^p K^p T^{p-1}\int_{T_{j-1}}^t \left|X_{\eta_n(s)}^{(n)} - X_s^{(n)}\right|^{p\beta} ds \\
&\quad + d^{\frac{p}{2}-1}6^{p-1}(1+\varepsilon)^p K^p T^p |s-\eta_n(s)|^{p\eta} \\
&\quad + d^{\frac{p}{2}-1}6^{p-1}\left|\int_0^t\int_0^\infty\int_{\mathbb{R}^d} K_i(s, X_{s^-}, X_{s^-}^{(n)}, r, z)\tilde{\mathcal{N}}(dz\times dr\times ds)\right|^p.
\end{aligned}$$

Since ε is arbitrary it can be chosen so that $c_p := d^{\frac{p}{2}}6^{p-1}\varepsilon^p < 1$. Now taking supremum and then expectation on both sides of the above inequality, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq u\leq t}\left|X_t - X_t^{(n)}\right|^p\right] \\
&\leq \frac{d^{\frac{p}{2}-1}6^{p-1}(\lambda\varepsilon)^p T^{p-1}}{(1-c_p)}\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_s - X_s^{(n)}\right|^p\right] ds \\
&+ \frac{d^{\frac{p}{2}-1}6^{p-1}(1+\varepsilon)^p K^p T^{p-1}}{(1-c_p)}\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_{\eta_n(s)}^{(n)} - X_s^{(n)}\right|^{p\beta}\right] ds \\
&+ \frac{d^{\frac{p}{2}-1}6^{p-1}(1+\varepsilon)^p K^p T^p}{(1-c_p)}|s-\eta_n(s)|^{p\eta} \\
&+ \frac{d^{\frac{p}{2}-1}6^{p-1}}{(1-c_p)}\mathbb{E}\left[\sup_{0\leq u\leq t}\left|\int_0^t\int_0^\infty\int_{\mathbb{R}^d} K_i(s, X_{s^-}, X_{s^-}^{(n)}, r, z)\tilde{\mathcal{N}}(dz\times dr\times ds)\right|^p\right]. \\
&= T^2 + T^3 + T^4 + T^5. \qquad (4.8)
\end{aligned}$$

To find estimates for the stochastic integral T^5, we use Doob's maximal inequality. Note that, in the constant coefficient case,

$$K(s,.,r,z) := H(s,x,r,z) - H(s,y,r,z) = 1_{[0,\ \sigma_0(s,z)]}(r)\left(\mathcal{T}_z u_i(s,x) - \mathcal{T}_z u_i(s,y)\right). \quad (4.9)$$

where, for any function f on $\mathbb{R}^d$, the operator $\mathcal{T}_z$ is defined by

$$\mathcal{T}_z f(s,x) = f(x+z) - f(x) \qquad \forall\ \ z\in\mathbb{R}^d.$$

Let us set

$$dA(s) := \int_{|z|\leq 1}\left(\mathcal{M}\left|\nabla\mathcal{T}_z u\right|(s,X_s) + \mathcal{M}\left|\mathcal{T}_z u\right|(s,X_s^{(n)})\right)^2\nu(dz).$$

Then using [[15], Lemma 5.2] and Minkowski inequality, there exists $c_2 > 0$ such that

$$\begin{aligned}\mathbb{E}\left[A(s)\right] &= \mathbb{E}\left[\int_{|z|\leq 1}\int_0^t \left(\mathcal{M}\left|\nabla \mathcal{T}_z u\right|(s,X_s) + \mathcal{M}\left|\mathcal{T}_z u\right|(s,X_s^{(n)})\right)^2 ds\nu(dz)\right] \\ &\leq c_2 \int_{|z|\leq 1} \|\nabla \mathcal{T}_z u\|^2_{L^\infty_{p'}(T)}\nu(dz) \leq C_3\|u\|_{L^\infty_T B^{\alpha+\beta}_{p',\infty}} \leq \varepsilon. \qquad (4.10)\end{aligned}$$

(see [15], page 36.) In the last inequality we have used Lemma 2.2. Thus applying Doob's maximal inequality, the hypothesis (H_1^σ), and using (4.10), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|T^5|^p\right] &= C_p\mathbb{E}\left[\left(\int_0^t\int_0^\infty\int_{|z|\leq 1}|K(s,X_{s^-},X^{(n)}_{s^-},r,z)|^2\tilde{\mathcal{N}}(dz,dr,dt)\right)^{\frac{p}{2}}\right] \\ &\leq C_p\mathbb{E}\left(\int_0^t\int_{|z|\leq 1}|\sigma(s,z)||\mathcal{T}_z u(s,X_{s^-}) - \mathcal{T}_z u(s,X^{(n)}_{s^-})|^2\nu(dz)ds\right)^{\frac{p}{2}} \\ &\leq C_p\mathbb{E}\left[\left(\kappa_2\int_{T_{j-1}}^t\int_{|z|\leq 1}|\mathcal{T}_z u(s,X_{s^-}) - \mathcal{T}_z u(s,X^{(n)}_{s^-})|^2\nu(dz)ds\right)^{\frac{p}{2}}\right] \\ &\leq C_p\mathbb{E}\left[\left(\int_0^t \kappa_2\varepsilon^2\left|X_s - X_s^{(n)}\right|^2 dA_s\right)^{\frac{p}{2}}\right] \\ &\leq C_p\left(\mathbb{E}\left[A_s\right]\right)^{\frac{p}{2}}\left(\mathbb{E}\left[\int_0^t \kappa_2\varepsilon^2\left|X_s - X_s^{(n)}\right|^2 ds\right]\right)^{\frac{p}{2}} \\ &\leq C_{p,T,\kappa_2,\varepsilon,b^p_{p'}}\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_u - X_u^{(n)}\right|^2\right]ds \qquad (4.11)\end{aligned}$$

Substituting (4.11) back into (4.8) and then using Lemma 2.5, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq u\leq t}\left|X_t - X_t^{(n)}\right|^p\right] &\leq \frac{d^{\frac{p}{2}-1}6^{p-1}(\lambda\varepsilon)^p}{(1-c_p)}\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_u - X_u^{(n)}\right|^p\right]ds \\ &\quad + \frac{d^{\frac{p}{2}-1}6^{p-1}(1+\varepsilon)^p K^p}{(1-c_p)}\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X^{(n)}_{\eta_n(s)} - X_s^{(n)}\right|^{p\beta}\right]ds \\ &\quad + \frac{d^{\frac{p}{2}-1}6^{p-1}(1+\varepsilon)^p (KT)^p}{(1-C_p)}\frac{1}{n^{p\eta}} \\ &\quad + C_{p,T,\kappa_2,\|b\|_p}\int_0^t \mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_u - X_u^{(n)}\right|^p\right]ds \\ &\leq C_2\mathbb{E}\left[\sup_{0\leq u\leq s}\left|X_u - X_u^{(n)}\right|^p\right]ds \\ &\quad + \frac{C_3}{n^{p\eta}} + \begin{cases}\frac{C_3}{n^{\frac{p\beta}{2}}} & if \qquad p\beta > 1 \\ \frac{C_3}{n^{\frac{p\beta}{\alpha}}} & if \qquad p\beta < \alpha \in (0,1).\end{cases}\end{aligned}$$

Finally, using mathematical induction on $j = 1, 2, ..., m$ and Gronwall's Lemma, one verifies that

$$\mathbb{E}\left[\sup_{0\le u\le t}\left|X_u - X_u^{(n)}\right|^p\right] \le \begin{cases} \frac{A_j}{n^{\frac{p\beta}{2}}} & if \qquad p\beta > 1 \\ \frac{A_j}{n^{\frac{p\beta}{\alpha}}} & if \qquad p\beta < \alpha \in (0,1). \end{cases}$$

where in each case $A_1 = C_2 e^{C_1 T}$ and $A_j := (C_0 A_{j-1} + C_2)e^{C_1 T}$ for all $j = 2, 3, ..., m$. Thus

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le T}\left|X_u - X_u^{(n)}\right|^p\Big] &\le \sum_{j=1}^{m}\Big[\sup_{T_{j-1}\le u\le T_j}\left|X_u - X_u^{(n)}\right|^p\Big] \\ &\le \left(\sum_{j=1}^{m} A_j\right)\begin{cases}\frac{1}{n^{\frac{p\beta}{2}}} & if \qquad p\beta > 1 \\ \frac{1}{n^{\frac{p\beta}{\alpha}}} & if \qquad p\beta < \alpha \in (0,1).\end{cases}\end{aligned}$$

The proof for the case where σ is space-dependent is similar, we omit. The proof is complete. □

4.2 Proof of Theorem 3.3 : As in Theorem 3.1, we prove the Theorem for the case where the function σ is independent of the space variable, i.e. $\sigma = \sigma_0(t, z)$. The proof of the case where σ is space-dependent is similar. Let us introduce two new variables Y_t and $\mathcal{T}_t(X_t^{(n)})$ defined by

$$Y_t := \mathcal{T}_t(X_t) = X_t + u(t, X_t), \qquad and \qquad \mathcal{T}_t(X_t^{(n)}) = X_t^{(n)} + u(t, X_t^{(n)}). \tag{4.12}$$

Substituting back into (4.6) and taking absolute value on both sides we have

$$\begin{aligned}&\left|Y_t^{(i)} - \mathcal{T}(X_t^{(n)})^i\right| \\ &\le \int_{T_{j-1}}^{t}\left|\tilde{b}(s, Y_s) - \tilde{b}(s, \mathcal{T}(X_s^{(n)})\right| ds + \int_0^t \left|b_i(\eta_n(s), X^{(n)}_{\eta_n(s)}) - b_i(s, X_s^{(n)})\right| ds \\ &+ \int_0^t \left|\nabla u_i(s, X_{s^-}^{(n)})\right|\left|b(\eta_n(s), X^{(n)}_{\eta_n(s)}) - b(s, X_s^{(n)})\right| ds \\ &+ \left|\int_0^t\int_0^\infty\int_{|z|\le 1} K(s, \mathcal{T}^{-1}(Y_{s^-}), \mathcal{T}^{-1}(\mathcal{T}(X_{s^-}^{(n)})), r, z)\tilde{\mathcal{N}}(dz \times dr \times ds)\right| \\ &+ \left|\int_0^t\int_0^\infty\int_{|z|> 1} K(s, \mathcal{T}^{-1}(Y_{s^-}), \mathcal{T}^{-1}(\mathcal{T}(X_{s^-}^{(n)}), r, z)\mathcal{N}(dz \times dr \times ds)\right|. \end{aligned}\tag{4.13}$$

We proceed to find a bound for each of the terms on the right side of (4.13). For the second term, we use Lemma 2.4 :

$$\int_0^t \left|\tilde{b}(s, Y_s) - \tilde{b}(s, \mathcal{T}(X_s^{(n)})\right| ds \le 2\varepsilon(\lambda + \kappa_2\nu(B_1^c))\int_0^t \left|Y_s - \mathcal{T}(X_s^{(n)})\right| ds. \tag{4.14}$$

The third and fourth terms are more challenging since, unlike in the case of Theorem 3.1, the coefficient b is not required to be Hölder continuous in x. To overcome this difficulty, we use an approximating sequence. Following [2], [15], consider $\rho \in C_0^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} dx = 1$. For $m \in \mathbb{N}$, define $\rho_m(t,x) := m^d \rho(t, mx)$ and

$$b_m(t,x) = \int_{\mathbb{R}^d} b(s,x)\rho_m(t-s, x-y)dy$$

Then

$$b_m \in H^{1,1}(\mathbb{R}^d) \cap C^1(\mathbb{R}^d), \quad b_m \longrightarrow b \ , \|b_m\|_{p'} \text{ and } \|\nabla b_m\|_{p'} \leq \|\nabla b\|_{p'}.$$

Using the Hardy-Littlewood function of the function of b_m, we have

$$|b_m(t,x) - b_m(t,y))| \leq C_d \, |x-y| \, (\mathcal{M}|\nabla b_m|(x) + \mathcal{M}|\nabla b_m|(y)) \leq C_{d,p'} \, \|\nabla b_m\|_{p'} \, |x-y|$$

for some constant $C_{d,p'} > 0$ (see [15]). Now since b is, by assumption, η-Hölder continuous in t, so is b_m. We have

$$\begin{aligned}
T_3 &:= \int_0^t \left| b(s, X_s^{(n)}) - b(\eta(s), X_{\eta(s)}^{(n)}) \right| ds \\
&\leq \int_0^t \left| b(s, X_s^{(n)}) - b_m(s, X_s^{(n)}) \right| ds + \int_{T_{j-1}}^t \left| b_m(s, X_s^{(n)}) - b_m(s, X_{\eta(s)}^{(n)}) \right| ds \\
&\quad + \int_0^t \left| b_m(s, X_{\eta_n(s)}^{(n)}) - b_m(\eta_(s), X_{\eta_n(s)}^{(n)}) \right| ds \\
&\quad + \int_0^t \left| b_m(\eta_n(s), X_{\eta_n(s)}^{(n)}) - b(\eta_n(s), X_{\eta_n(s)}^{(n)}) \right| ds \\
&\leq KT \left(\frac{T}{n}\right)^\eta + C_{d,p} K \, \|\nabla b_m\|_{p'} \int_0^t \left| X_s^{(n)} - X_{\eta(s)}^{(n)} \right| ds + C \, \|b_m - b\|_{p'} \\
&\leq KT \left(\frac{T}{n}\right)^\eta + C_{d,p} K \, \|\nabla b\|_{p'} \int_0^t \left| X_s^{(n)} - X_{\eta(s)}^{(n)} \right| ds, \qquad (4.15)
\end{aligned}$$

where we applied Hölder's inequality and took the limit as $m \longrightarrow \infty$ in the last two inequalities. Next, we substitute the inequalities (4.14) and (4.15), back into the inequality (4.13) and then use Jensen and Hölder's inequalities to form the L^p-norm of

the error. We have

$$
\begin{aligned}
&\left|Y_t - \mathcal{T}(X_t^{(n)})\right|^p \\
&= \left(\sum_{i=1}^d \left|Y_t^i - \mathcal{T}_t(X_t^{(n)})^i\right|^2\right)^{\frac{p}{2}} = d^{\frac{p}{2}-1}5^{p-1}\sum_{i=1}^d \left|Y_t^i - \mathcal{T}_t(X_t^{(n)})^i\right|^p \\
&\le d^{\frac{p}{2}}5^{p-1}(\varepsilon K_{\lambda,\sigma,\nu})^p T^{p-1}\int_{T_{j-1}}^t \left|Y_s - \mathcal{T}(X_s^{(n)})\right|^p ds \\
&+ d^{\frac{p}{2}-1}5^{p-1}(1+\varepsilon)^p K^p T^p \left(\frac{T}{n}\right)^{p\eta} \\
&+ d^{\frac{p}{2}}5^{p-1}(1+\varepsilon)^p K^p T^{p-1} C_{d,p}\left\|\nabla b\right\|_{p'}^p \int_0^t \left|X_s^{(n)} - X_{\eta(s)}^{(n)}\right|^p ds \\
&+ d^{\frac{p}{2}-1}5^{p-1}\left|\int_0^t\int_0^\infty\int_{|z|\le 1} K(s,\mathcal{T}^{-1}(Y_{s^-}),\mathcal{T}^{-1}(\mathcal{T}(X_{s^-}^{(n)})),r,z)\tilde{\mathcal{N}}(dz\times dr\times ds)\right|^p \\
&+ d^{\frac{p}{2}-1}5^{p-1}\left|\int_0^t\int_0^\infty\int_{|z|> 1} K(s,\mathcal{T}^{-1}(Y_{s^-}),\mathcal{T}^{-1}(\mathcal{T}(X_{s^-}^{(n)})),r,z)\mathcal{N}(dz\times dr\times ds)\right|^p \\
&\le T^1 + T^2 + T^3 + T^4 + T^5 + T^6. \qquad (4.16)
\end{aligned}
$$

To control the two stochastic integrals, we apply Doob's maximal inequality to the terms T^5 and T^6 with $K(s,x,y,r,z)$ defined in (4.9). We have

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|T^5|^p\right] \\
&:= \mathbb{E}\left[\sup_{0\le t\le T}\left|\int_{T_{j-1}}^t\int_0^\infty\int_{|z|\le 1} K(s,\mathcal{T}^{-1}(Y_{s^-}),\mathcal{T}^{-1}(\mathcal{T}(X_{s^-}^{(n)}),r,z)\tilde{\mathcal{N}}(dz,dr,dt)\right|^p\right] \\
&\le C_p\mathbb{E}\left[\left(\int_{T_{j-1}}^t\int_{|z|\le 1}\sigma_0(s,z)\left|\mathcal{T}^{-1}(Y_{s^-}) - \mathcal{T}^{-1}(\mathcal{T}(X_{s^-}^{(n)}))\right|^2\nu(dz)ds\right)^{\frac{p}{2}}\right] \\
&\le 4C_p\varepsilon\mathbb{E}\left[\left(\int_{T_{j-1}}^t\int_{|z|\le 1}\sigma_0(s,z)\left|Y_{s^-} - \mathcal{T}(X_{s^-}^{(n)})\right|^2\nu(dz)ds\right)^{\frac{p}{2}}\right] \\
&\le 4\kappa_2\nu(B_1))^{\frac{p}{2}}T^{\frac{p}{2}-1}\mathbb{E}\left[\int_0^t \kappa_2\left|Y_{s^-} - \mathcal{T}(X_{s^-}^{(n)})\right|^p ds\right], \qquad (4.17)
\end{aligned}
$$

Similarly, one obtains

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}|T^6|^p\right]\\
&:=\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t\int_0^\infty\int_{|z|>1}K(s,\mathcal{T}^{-1}(Y_{s^-}),\mathcal{T}^{-1}(\mathcal{T}(X^{(n)}_{s^-})),r,z)\mathcal{N}(dz,dr,dt)\right|^p\right]\\
&\le C_{p,T,\sigma_0,\nu,\varepsilon}\mathbb{E}\left[\int_0^t\left|Y_{s^-}-\mathcal{T}(X^{(n)}_{s^-})\right|^p ds\right]. \qquad (4.18)
\end{aligned}$$

Thus, taking supremum and then expectation on both sides of the inequality (4.16) and using the inequalities (4.17) and (4.18), we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{T_{j-1}\le u\le t}\left|Y_u-\mathcal{T}(X^{(n)}_u)\right|^p\right]\\
&\le d^{\frac{p}{2}}5^{p-1}(\varepsilon K_{\lambda,\sigma,\nu})^pT^{p-1}\int_0^t\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\left|Y_u-\mathcal{T}(X^{(n)}_u)\right|^p\right]ds\\
&+d^{\frac{p}{2}-1}5^{p-1}K^pT^p\left(\frac{T}{n}\right)^{p\eta}\\
&+d^{\frac{p}{2}-1}5^{p-1}T^{p-1}C_{d,p}K^p\left\|b\right\|_r^p\int_0^t\mathbb{E}\left[\sup_{0\le u\le s}\left|X^{(n)}_u-X^{(n)}_{\eta(u)}\right|^p\right]ds\\
&+d^{\frac{p}{2}-1}5^{p-1}C_{p,T,\sigma_0,\nu,\varepsilon}\int_0^t\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\left|Y_{u^-}-\mathcal{T}(X^{(n)}_{u^-})\right|^p ds\right] \qquad (4.19)
\end{aligned}$$

Lastly, using Lemma 2.4, we have

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\le u\le t}\left|Y_u-\mathcal{T}(X^{(n)}_u)\right|^p\right] &\le C_2\int_0^t\mathbb{E}\left[\sup_{0\le u\le s}\left|Y_u-\mathcal{T}(X^{(n)}_u)\right|^p\right]ds+C_3\frac{1}{n^{p\eta}}\\
&\quad+\begin{cases}\frac{C_2}{n^{\frac{p}{2}}} & if \quad p>1\\ \frac{C_2}{n^{\frac{p}{\alpha}}} & if \quad p<\alpha\in(0,1)\end{cases}\\
&\le C_1\int_0^t\mathbb{E}\left[\sup_{T_{j-1}\le u\le s}\left|Y_u-\mathcal{T}(X^{(n)}_u)\right|^p\right]ds\\
&\quad+\begin{cases}\frac{C_2}{n^{\frac{p}{2}\wedge p\eta}} & if \quad p>1\\ \frac{C_2}{n^{\frac{p}{\alpha}\wedge p\eta}} & if \quad p<\alpha\in(0,1).\end{cases}
\end{aligned}$$

Finally, using mathematical induction on $j=1,2,...,m$ and Gronwall's Lemma, one verifies that

$$\mathbb{E}\left[\sup_{T_{j-1}\le u\le t}\left|Y_u-\mathcal{T}(X^{(n)}_u)\right|^p\right] \le \begin{cases}\frac{A_j}{n^{\frac{p}{2}\wedge p\eta}} & if \quad p>1\\ \frac{A_j}{n^{\frac{p}{\alpha}\wedge p\eta}} & if \quad p<\alpha\in(0,1).\end{cases}$$

where in each case $A_1 = C_2 e^{C_1 T}$ and $A_j := (C_0 A_{j-1} + C_2)e^{C_1 T}$ for all $j = 2, 3, ..., m$, completes the proof. Thus

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le u\le T}\Big|Y_u - \mathcal{T}(X_u^{(n)})\Big|^p\Big] &\le \sum_{j=1}^{m}\Big[\sup_{T_{j-1}\le u\le T_j}\Big|Y_u - \mathcal{T}(X_u^{(n)})\Big|^p\Big] \\ &\le \left(\sum_{j=1}^{m} A_j\right)\begin{cases}\frac{1}{n^{\frac{p}{2}\wedge p\eta}} & if \qquad p>1\\ \frac{1}{n^{\frac{p}{\alpha}\wedge p\eta}} & if \qquad p<\alpha\in(0,1).\end{cases}\end{aligned}$$

We poved that

$$\mathbb{E}\Big[\sup_{0\le u\le T}\Big|Y_u - \mathcal{T}(X_u^{(n)})\Big|^p\Big] \le \begin{cases}\frac{C}{n^{\frac{p}{2}\wedge p\eta}} & if \qquad p>1\\ \frac{C}{n^{\frac{p}{\alpha}\wedge p\eta}} & if \qquad p<\alpha\in(0,1).\end{cases}$$

The result follows from definitions (4.12) and Lemma 2.6.

Remark 4.1 : We note that in the case where the function σ is space-dependent, the constant C in the inequalities corresponding to the last three terms in (4.6) involve $\|\varrho\|^p_{L^\infty_T B^0_{q,\infty}}$. To see this, apply the Doob's inequality to the sixth term in (4.6) (for example), and use the hypothesis (H_2^σ), [[15], Lemma 5.2] and Minkowski inequality and Hölder's inequality.

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